Screening and vertex operators for u(n)

1991 J. Phys. A: Math. Gen. 24 1393

(http://iopscience.iop.org/0305-4470/24/7/016)

J. Phys. A: Math. Gen. **24** (1991) 1393–1425. Printed in the UK

Screening and vertex operators for u(n)

R Le Blanc†, K T Hecht‡ and L C Biedenharn§

† Rutherford Physics Building, McGill University, Montréal, Province of Québec, Canada H3A 2T8

‡ Department of Physics, University of Michigan, Ann Arbor, MI 48109, USA

§ Department of Physics, Duke University, Durham, NC 27706, USA

Received 13 August 1990, in final form 21 November 1990

Abstract. Screening and vertex operators for u(n) are given in terms of generators of the group left action on the flag manifold U($n-1$)/SU(n). Their matrix elements in a basis of holomorphic sections of a vector bundle are computed.

1. Introduction

The construction of solutions for two-dimensional conformal field theory is a problem of great current interest. It can be argued that two-dimensional conformal field theory and Wess–Zumino–Witten (WZW) theories are intimately related to the representation theory of the infinite-dimensional Virasoro and affine Kac–Moody algebras respectively. In this spirit, Bouwknegt *et al* (1990) have generalized results from Bernstein *et al* (1971), Kostant (1974), Želobenko (1973) and Kempf (1978) concerning the construction of irreducible modules of compact finite-dimensional Lie algebras in Fock spaces, to the construction of irreducible modules of affine algebras. Such constructions have much in common with the geometrical construction of Lie algebra modules in terms of holomorphic sections of a line bundle over an appropriate flag manifold.

The geometric approach to representation theory yields the celebrated Borel–Weil–Bott theorem (Bott 1957) which makes explicit the geometrical origin of Weyl's character formula (see also the recent partial rederivations of the theorem by physicists (Stone 1989, Alvarez *et al* 1990)). In this framework, Weyl's character formula is regarded as an alternating sum of traces over Fock spaces. The geometric approach has also been favoured in the physics literature, albeit under various guises. A non-exhaustive list would necessarily include: the coherent state approach to representation theory of Lie algebras (Perelomov 1986), which parallels the standard holomorphic line bundle construction; the more recent vector coherent state approach to representation theory of Lie (super)algebras (Rowe 1984, 1985, Rowe *et al* 1988, Deenen and Quesne 1984, Quesne 1986, Castaños *et al* 1985, Hecht 1987, Le Blanc and Rowe 1988, 1989, 1990), a generalization of ordinary coherent state theory which parallels the holomorphic vector bundle construction (Bott 1957, Griffiths and Schmid 1969); geometric quantization (Kostant 1970, 1977, Kirillov 1976, Woodhouse 1980, Guillemin and Sternberg 1982); boson expansion theories (Dobaczewski 1981, 1982, Klein and Marshalek 1990); free-field approach to the representation theory of the Virasoro and Kac–Moody algebras (Feigen and Frenkel 1990, Ito and Kazama 1989). These constructions share a common problem: the highest weight Fock space modules are, in general, not irreducible, nor completely reducible. A resolution (identification of the irreducible sub-modules) of the Fock space modules can nevertheless be achieved upon introduction of a new concept, that of 'screening' operators.

Screening operators have recently received much attention in the representation theory of infinite-dimensional algebras (Dotsenko and Fateev 1984, 1985, Bouwknegt *et al* 1990). They are used to construct invariant homomorphisms and singular vectors of non-fully reducible modules. Bouwknegt *et al* have underlined the fact that, in a Lie algebra context, the concept of screening operators stands for that of quasi-invariant differential operators (Kostant 1974). They also have alluded to the fact that explicit expressions for 'vertex' (equivariant tensor) operators, useful for the derivation of 'fusion' (Kronecker product) rules, can be given in terms of screening operators.

The primary focus of this paper is to elaborate on the concept of vertex operators in a Lie algebra context, although we remark that the present construction can be generalized to the quantum groups (Biedenharn 1990). Combining the strength of the vector coherent state approach and of the vertex operator formalism, we obtain analytical results concerning the structure of vertex operators. (Due to the relative ease with which one can construct Gel'fand bases for the unitary group (Louck 1970, Hecht *et al* 1987), we have restricted explicit computations to u(n) with special emphasis on u(3).) The *equivariance condition (2.6) is used for the explicit construction* of equivariant tensor operators in geometrical terms (sections 3 and 4). Their structure is discussed (section 4). In particular, the structure of the elementary, totally symmetric and octet u(3) vertex operators now have a canonical form which, for the first time, contains explicitly the hitherto abstract canonical upper pattern (section 4.5).

2. Vertex operators for Lie algebras

2.1. Fock space realization of Lie algebras

In this section, we review the results compiled by Bouwknegt *et al* (1990) concerning the construction of irreducible modules of finite-dimensional Lie algebras in Fock spaces as they provide the theoretical setting for the present study.

A realization for a simple finite-dimensional Lie algebra g of rank l can be given in terms of linear differential operators on the space of holomorphic sections of a line bundle over the flag manifold $\mathrm{B}_-\backslash\mathrm{G}$ parametrized by complex coordinates z. This bundle is determined by a character $\chi_\Lambda : \mathrm{B}_- \to \mathbf{C}^*$ of the Borel subgroup B_-. The sections can be identified with functions (coherent states)

$$\psi_\Lambda(z) = \langle\Lambda|\exp(z)|\psi\rangle \qquad z = \sum_{\alpha\in\Delta_+} z_\alpha e_\alpha \tag{2.1}$$

satisfying

$$\langle\Lambda|b\exp(z)|\psi\rangle = \chi_\Lambda(b)\langle\Lambda|\exp(z)|\psi\rangle \qquad \forall b \in \mathrm{B}_-.$$

The group G acts as a transformation group on the flag manifold by right multiplication. This induces a transformation on the coherent states and, consequently, a representation σ_Λ of the Lie algebra g in terms of linear differential operators:

$$\sigma_\Lambda(X)\langle\Lambda|\exp(z)|\psi\rangle \equiv \langle\Lambda|\exp(z)X|\psi\rangle \qquad \forall X \in \mathrm{g}. \tag{2.2}$$

For finite-dimensional representations of $\mathbf{g}$, it is sufficient to restrict the function space to polynomials in the complex coordinates in which case one can interpret the module as the Fock space F_Λ of a finite set of harmonic oscillators. There will be as many oscillators as there are positive roots for $\mathbf{g}$ as indicated in (2.1).

Information concerning the highest weight state can be given more explicitly in terms of pairs of momentum–position operators (p_i, q_i) with commutation relations

$$[p_i, q_j] = -\mathrm{i}\delta_{ij}.$$

The Fock space F_Λ associated with $\Lambda \in \mathbf{h}^*$ (where $\mathbf{h}^*$ is the dual space of the Cartan subalgebra $\mathbf{h} \subset \mathbf{g}$) is identified with $\mathrm{Pol}(z) \otimes \mathbf{C}_\Lambda$, where $\mathbf{C}_\Lambda$ is the one-dimensional space obtained from the highest weight state defined by $|\Lambda\rangle = \mathrm{e}^{\mathrm{i}\Lambda\cdot q}|0\rangle$. One thus has

$$\sigma_\Lambda(h_i)|\Lambda\rangle = p_i|\Lambda\rangle = \Lambda_i|\Lambda\rangle$$

where Λ_i denotes the components of Λ with respect to an orthonormal basis in $\mathbf{h}^*$. The translation operators $\mathrm{e}^{\mathrm{i}\Lambda\cdot q}$ span the 'model space' for $\mathbf{g}$ (Gel'fand and Zelevinsky 1985).

The Fock space realization described previously is not irreducible nor is it completely reducible. There are many ways to characterize the subspace of F_Λ corresponding to the irreducible module L_Λ of highest weight Λ. In particular, this can be done elegantly in terms of screening operators (Bouwknegt *et al* 1990). To do this, one must look at the left action of G.

Using left multiplication, one induces a representation ρ_Λ

$$\rho_\Lambda(X)\langle\Lambda|\exp(z)|\psi\rangle \equiv \langle\Lambda|(-X)\exp(z)|\psi\rangle \qquad \forall X \in \mathbf{n}_+ \tag{2.3}$$

for the raising subalgebra $\mathbf{n}_+$ of $\mathbf{g}$, commuting with, but isomorphic to, $\sigma_\Lambda(\mathbf{n}_+)$. Screening operators

$$s_i = \rho(e_{\alpha_i})\mathrm{e}^{\mathrm{i}\alpha_i\cdot q}$$

are then introduced such that the irreducible module L_Λ is given by

$$L_\Lambda = \{v \in F_\Lambda|(s_i)^{l_i}\cdot v = 0, \quad i = 1, 2, \ldots, l, \quad l_i = \langle\Lambda, h_i\rangle + 1\}. \tag{2.4}$$

This description of the irreducible module L_Λ is equivalent to the Borel–Weil description of a module in terms of holomorphic sections of a line bundle.

Given a simple Lie algebra $\mathbf{g}$, the Weyl group W of $\mathbf{g}$ is generated by the reflections r_i by the simple roots α_i of $\mathbf{g}$. Every element $w \in$ W can be written in the form $r_{i_1}r_{i_2}\cdots r_{i_n}$, and the length $l(w)$ is defined as the minimal number of reflections r_i required to construct w. Denote $W^{(k)} = \{w \in \mathrm{W}|l(w) = k\}$. A shifted action of W on $\Lambda \in \mathbf{h}^*$ is defined by

$$w * \Lambda = w(\Lambda + \rho) - \rho$$

with ρ half the sum of the positive roots of $\mathbf{g}$. For w_1, $w_2 \in$ W, we write $w_1 \leftarrow w_2$ if $w_1 = r_\alpha w_2$ for some $\alpha \in \Delta_+$, and also $l(w_1) = l(w_2) + 1$. A partial (Bruhat) ordering can be defined on W: $w \prec w'$ if and only if there exists $w_1, w_2, \ldots, w_k \in$ W such that

$$w \leftarrow w_1 \leftarrow w_2 \leftarrow \ldots \leftarrow w_k \leftarrow w'.$$

The operators $Q^{(i)}_{l_i} \equiv (s_i)^{l_i}$ are g-algebra invariants when acting on F_Λ. We have the action

$$Q^{(i)}_{l_i} : F_\Lambda \to F_{r_i * \Lambda}$$

that is, the Qs are intertwining operators. An intertwining operator $Q : \mathrm{V} \to \mathrm{W}$ is a homomorphism $\mathrm{V} \to \mathrm{W}$ of two g-modules commuting with the g-action on V and W. The set of such intertwining operators is denoted $\mathrm{Hom}_{\mathcal{U}(\mathbf{g})}(\mathrm{V}, \mathrm{W})$ where $\mathcal{U}(\mathbf{g})$ is the enveloping algebra of $\mathbf{g}$.

Let

$$M_\Lambda = \mathcal{U}(\mathbf{g}) \cdot v_\Lambda$$

with v_Λ a highest weight vector, be the usual Verma module. There exists a one-to-one correspondence between singular vectors in the Verma module, that is, states v such that $\mathbf{n}_+ \cdot v = 0$, and invariant intertwiners on F_Λ. For simple Lie algebras and finite-dimensional representations, these singular vectors v_w are in one-to-one correspondence with elements in the Weyl group and occur for the weights $w * \Lambda$, $w \in \mathrm{W}$. Moreover, $v_{w'} \in M_{w*\Lambda}$ if and only if $w' \preceq w$. In this case, a homomorphism

$$Q_{w,w'} \in \mathrm{Hom}_{\mathcal{U}(\mathbf{g})}(F_{w*\Lambda}, F_{w'*\Lambda})$$

exists and a complex of Fock modules can be built in terms of these homomorphisms such that

$$0 \to F^{(0)}_\Lambda \xrightarrow{d^{(0)}} F^{(1)}_\Lambda \xrightarrow{d^{(1)}} \dots \xrightarrow{d^{(s-1)}} F^{(s)}_\Lambda \to 0$$

where s is the order of Δ_+, and

$$F^{(i)}_\Lambda = \oplus_{w \in W^{(i)}} F_{w*\Lambda}.$$

Explicit expressions for the su(3) *nilpotent operators* $d(Q)$ can be found in Bouwknegt *et al* (1990).

With $H^i(d) = \mathrm{Ker}\, d^{(i)}/\mathrm{Im}\, d^{(i-1)}$, one has

$$H^i(d) = \begin{cases} L_\Lambda & i = 0 \\ 0 & \text{otherwise.} \end{cases} \tag{2.5}$$

2.2. Vertex operators

Abstractly, a shift tensor operator $\mathcal{T}_{\Lambda_2}$ is a tensor operator which maps the Hilbert space $\mathcal{H}_{\Lambda_1}$ carrying the irrep Λ_1 of $\mathbf{g}$ to the Hilbert space $\mathcal{H}_{\Lambda_3}$ and to this one only. Such an operator is said to have 'good shift properties'. Vertex operators V are simply oscillator realizations and generalizations of shift tensor operators, that is, they map

$$V_{\Lambda_2}(\Lambda_1, \Lambda_3) : F_{\Lambda_1} \to F_{\Lambda_3}$$

and, more precisely, they map the irreducible submodules $L_{\Lambda_1} \to L_{\Lambda_3}$. We shall designate by $V^{(i)}$ the set of vertex operators which maps

$$V^{(i)} : F^{(i)}_{\Lambda_1} \to F^{(i)}_{\Lambda_3}.$$

The equivariance condition

$$d^{(i)}V^{(i)} = V^{(i+1)}d^{(i)}$$

must be required for the vertex operators to have well-defined shift properties on the irreducible submodules. It is equivalent to restrict to the equivariance condition

$$d^{(0)}V^{(0)} = V^{(1)}d^{(0)}$$

or, in more explicit terms, to

$$\begin{aligned} &Q^{(i)}_{l_i^{\Lambda_3}} V_{\Lambda_2}(\Lambda_1, \Lambda_3) = V_{\Lambda_2}(r_i * \Lambda_1, r_i * \Lambda_3) Q^{(i)}_{l_i^{\Lambda_1}} \\ &l_i^{\Lambda} = \langle \Lambda, h_i \rangle + 1 \qquad i = 1, 2, \ldots, l \end{aligned} \tag{2.6}$$

which is a key equation for the following.

Vertex operators for finite-dimensional Lie algebras are relatively easy to construct (Bouwknegt *et al* 1990). A vertex operator V_{Λ_2} with maximal weight and shift is given by the translation operator $e^{i\Lambda_2 \cdot q}$. All other highest weight vertex operators with lower shifts are obtained by multiplying $e^{i\Lambda_2 \cdot q}$ by terms with an appropriate number of screening charges. In section 3, we carry out this construction in its entirety for u(2). In section 4, important classes of vertex operators will be constructed for u(3) for which the u(2) results shall be needed.

2.3. The u(n) algebra

For notational simplicity, it is easier to work with the non-simple Lie algebra $u(n) = u(1) \oplus su(n)$. The $u(n)$ Lie algebra is defined through the commutator algebra

$$[E_{ij}, E_{kl}] = \delta_{jk} E_{il} - \delta_{il} E_{kj} \qquad 1 \le i, j, k, l \le n. \tag{2.7}$$

Simple roots for its $su(n)$ subalgebra are given by

$$e_{\alpha_i} = e_i = E_{i,i+1} \qquad 1 \le i \le n-1. \tag{2.8}$$

In root space, we set

$$\alpha_i = e_i - e_{i+1} \tag{2.9}$$

with Euclidean scalar product

$$(e_i, e_j) = \delta_{ij}. \tag{2.10}$$

The set of positive roots is given by

$$\Delta_+ = \{(e_i - e_j), \qquad i < j\}$$

the set of negative roots by

$$\Delta_- = \{(e_i - e_j), \qquad i > j\}.$$

The corresponding generators generate the nilpotent subalgebras $\mathbf{n}_+$ and $\mathbf{n}_-$ respectively. For su(n), half the sum of the positive roots is given by

$$\rho = \frac{1}{2}\left[\sum_{k=1}^{n}(n-2k+1)e_k\right]. \tag{2.11}$$

The Cartan subalgebra for u(n) comprises the n weight operators E_{ii}, $1 \leq i \leq n$. For su(n), it is spanned by the $n-1$ weight operators $h_i = E_{ii} - E_{i+1,i+1}$.

Irreducible representations for $u(n)$ are labelled by the n entries m_{in} of the array $[\boldsymbol{m}_n] = [m_{1n}m_{2n}\dots m_{nn}]$ where the differences $m_{in} - m_{i+1,n}$ are non-increasing integers. One has that

$$E_{ii}|[\boldsymbol{m}_n]\mathrm{hw}\rangle = m_{in}|[\boldsymbol{m}_n]\mathrm{hw}\rangle.$$

Equivalently, these irreps are labelled by the highest weights

$$\boldsymbol{m}_n = \sum_{i=1}^{n} m_{in}\boldsymbol{e}_i.$$

States in these irreps are labelled by the usual Gel'fand patterns (Louck 1970). In our model space,

$$|[\boldsymbol{m}_n]\mathrm{hw}\rangle = \exp\left(\mathrm{i}\sum_{i=1}^{n} m_{in}q_i\right)|0\rangle.$$

We shall consistently use in the following the so-called partial hooks p_{ij} defined by

$$p_{ij} = m_{ij} + j - i \tag{2.12}$$

(not to be confused with the momentum operators p_i). The importance of the partial hooks lies in the fact that the Weyl reflections acting on the irrep labels have simple expressions in terms of the partial hooks: the Weyl reflection r_i interchanges p_{in} and $p_{i+1,n}$, that is, the Weyl group for su(n) acts on the partial hooks by permutations.

2.4. Unit tensor operators for u(n)

Unit shift tensor operators for u(n) can be given a canonical matrix realization as follows. Define a unit tensor operator T by the symbol

$$\left\langle \begin{matrix} (\Gamma)_{n-1} \\ [\boldsymbol{m}_n^{(t)}] \\ (m^{(t)})_{n-1} \end{matrix} \right\rangle \tag{2.13}$$

where

$$\left(\begin{matrix} [\boldsymbol{m}_n^{(t)}] \\ (m^{(t)})_{n-1} \end{matrix} \right)$$

is the Gel'fand–Weyl pattern associated with the vector $(m^{(t)})_n$ of the irrep $[m_n^{(t)}]$, and $(\Gamma)_{n-1}$ is an operator (inverted) pattern which canonically resolves all multiplicity (Louck and Biedenharn 1970). Such a unit tensor operator $\mathcal{T}$ in u(n) has an action on a generic state vector that effects a change in the u(n) representation labels, that is, $\mathcal{T}: [m_n] \to [m_n] + [\Delta(\Gamma)]$, where $[\Delta(\Gamma)] = [\Delta_1(\Gamma), \Delta_2(\Gamma), \ldots, \Delta_n(\Gamma)]$ denotes the label shifts $\Delta_i(\Gamma)$ (Biedenharn and Louck 1968, Louck 1970) and the sum is done componentwise ($m_{in} \to m_{in} + \Delta_i(\Gamma)$). The matrices of the unit tensor operator $\mathcal{T}$ are then $u(n)$ vector addition (Wigner) coefficients:

$$\left\langle \begin{matrix} [m_n^{(f)}] \\ (m^{(f)})_{n-1} \end{matrix} \right| \left\langle \begin{matrix} (\Gamma)_{n-1} \\ [m_n^{(t)}] \\ (m^{(t)})_{n-1} \end{matrix} \right\rangle \left| \begin{matrix} [m_n^{(i)}] \\ (m^{(i)})_{n-1} \end{matrix} \right\rangle \tag{2.14}$$

where $[m^{(f)}] = [m^{(i)}] + [\Delta(\Gamma)]$. (More details of this standard construction can be found in Louck and Biedenharn 1970.)

The concept of projective operators (Louck and Biedenharn 1970), also needed in the following, has its origin in the observation that a tensor operator in u(n) is, at the same time, a tensor operator in the subgroup u($n-1$). Assuming that all u($n-1$) unit tensor operators are known, we may expand the u(n) unit tensor operator

$$\left\langle \begin{matrix} (\Gamma)_{n-1} \\ [m_n^{(t)}] \\ (m^{(t)})_{n-1} \end{matrix} \right\rangle = \sum_{(\gamma)_{n-2}} \begin{bmatrix} (\Gamma)_{n-1} \\ [m_n^{(t)}] \\ (\gamma)_{n-1} \end{bmatrix} \left\langle \begin{matrix} (\gamma)_{n-2} \\ [m_{n-1}^{(t)}] \\ (m^{(t)})_{n-2} \end{matrix} \right\rangle \qquad \gamma_{i,n-1} = m_{i,n-1}^{(t)} \tag{2.15}$$

where:

(i) the object in the square brackets on the right-hand side denotes a unit projective operator, on the u(n) : u($n-1$) space (thus an u($n-1$) invariant operator), and

(ii) $\langle [m_{n-1}^{(t)}] \rangle$ is a u($n-1$) unit tensor operator with upper operator pattern $(\gamma)_{n-2}$. The left-hand side of (2.15) operates on u(n) vectors $|(m)_n\rangle$ where $(m)_n = (m_{ij})$, $1 \le i,j \le n$, is an arbitrary Gel'fand–Weyl u(n) pattern, whilst the right-hand side has $\langle [m_{n-1}^{(t)}] \rangle$ acting on vectors $|(m)_{n-1}\rangle$ of the u($n-1$) subgroup with $(m)_{n-1} = (m_{ij})$, $1 \le i,j \le n-1$; therefore, the projective operator acts on the factor space u(n) : u($n-1$). To be fully explicit, let us state that the matrix elements of the projective operators (isoscalar factors) take the form

$$\left\langle \begin{matrix} [m_n + \Delta(\Gamma)] \\ [m_{n-1} + \Delta(\gamma)] \end{matrix} \right| \begin{bmatrix} (\Gamma)_{n-1} \\ [m_n^{(t)}] \\ (\gamma)_{n-1} \end{bmatrix} \left| \begin{matrix} [m_n] \\ [m_{n-1}] \end{matrix} \right\rangle \tag{2.16}$$

where the u($n-1$) shifts $\Delta_i(\gamma)$ are defined similarly to the u(n) shift $\Delta_i(\Gamma)$ shifts.

3. Vertex operators for u(2)

The realization of the Lie algebra u(2) in terms of linear differential operators is well known. The representation σ is defined by

$$\sigma(X)\langle hw| \exp(z_{12}E_{12})|\psi\rangle = \langle hw| \exp(z_{12}E_{12})X|\psi\rangle \qquad X \in \mathrm{u}(2). \tag{3.1}$$

From this, one easily obtains

$$\begin{aligned}\sigma(E_{12}) &= \partial_{12}\\ \sigma(E_{11}) &= p_1 - z_{12}\partial_{12}\\ \sigma(E_{22}) &= p_2 + z_{12}\partial_{12}\\ \sigma(E_{21}) &= (p_1 - p_2)z_{12} - z_{12}^2\partial_{12}\end{aligned} \tag{3.2}$$

with the complex variable z_{12} parametrizing the flag manifold $\mathrm{U}(1)\backslash\mathrm{SU}(2)$. The highest weight vector $|hw\rangle$ is written in terms of two pairs of momentum–position operators (p_i, q_i) with

$$[p_i, q_j] = -\mathrm{i}\delta_{ij} \qquad 1 \leq i, j \leq 2$$

such that

$$|hw\rangle = \left| \begin{matrix} m_{12} & & m_{22} \\ & m_{12} & \end{matrix} \right\rangle = \mathrm{e}^{\mathrm{i}(m_{12}q_1 + m_{22}q_2)}|0\rangle.$$

The action of the algebra (3.2) is unitary on the Fock space basis of mononomials

$$\left\langle z \left| \begin{matrix} m_{12} & & m_{22} \\ & m_{11} & \end{matrix} \right.\right\rangle = K \begin{pmatrix} m_{12} & & m_{22} \\ & m_{11} & \end{pmatrix} \frac{z_{12}^{m_{12}-m_{11}}}{\sqrt{(m_{12} - m_{11})!}} \left| \begin{matrix} m_{12} & & m_{22} \\ & m_{12} & \end{matrix} \right\rangle \tag{3.3}$$

with the betweeness condition $m_{12} \leq m_{11} \leq m_{22}$ for the u(2) Gel'fand pattern. The unitarizing coefficient K is given by

$$K \begin{pmatrix} m_{12} & & m_{22} \\ & m_{11} & \end{pmatrix} = \left[\frac{(m_{12} - m_{22})!}{(m_{11} - m_{22})!}\right]^{1/2} \tag{3.4}$$

(see also equation (4.5)).

The left action for the raising algebra is given by

$$\rho(E_{12})\langle hw| \exp(z_{12}E_{12})|\psi\rangle = \langle hw|(-E_{12}) \exp(z_{12}E_{12})|\psi\rangle$$

that is

$$\rho(E_{12}) = -\partial_{12} \tag{3.5}$$

and the unique screening charge for u(2) is given by

$$s_{12} = \mathrm{e}^{-\mathrm{i}(q_1 - q_2)}\rho(E_{12}). \tag{3.6}$$

Under the algebra (3.2), the screening charge has null weight and maps the irrep $[m_2]$

$$s_{12} : [m_{12}m_{22}] \quad \rightarrow \quad [m_{12} - 1, m_{22} + 1].$$

The elementary unnormalized vertex operators in tensor operator notation are given by

$$\left\langle \begin{matrix} & 1 & \\ 1 & & 0 \\ & 1 & \end{matrix} \right\rangle_{\mathrm{un}} = \mathrm{e}^{\mathrm{i}q_1} \tag{3.7a}$$

$$\left\langle \begin{matrix} & 0 & \\ 1 & & 0 \\ & 1 & \end{matrix} \right\rangle_{\mathrm{un}} = \mathrm{e}^{\mathrm{i}q_1} s_{12}. \tag{3.7b}$$

It is easily verified that their su(2) reduced matrix elements are given by 1 and

$$[(m_{12}-m_{22}+1)(m_{12}-m_{22})]^{1/2} = [(p_{22}-p_{12}+1)(p_{22}-p_{12})]^{1/2} \tag{3.8}$$

respectively when these operators act on the irrep $[m_{12}m_{22}]$.

A generic u(2) unnormalized vertex operator is given by

$$\left\langle \begin{matrix} & \Gamma_{11} & \\ m_{12} & & 0 \\ & m_{12} & \end{matrix} \right\rangle_{\mathrm{un}} = \mathrm{e}^{\mathrm{i}m_{12}q_1} s_{12}^{m_{12}-\Gamma_{11}} \tag{3.9}$$

with shifts $\Delta(\Gamma) = [m^{(f)}] - [m^{(i)}]$, that is

$$\Delta_1(\Gamma) = \Delta_1 = \Gamma_{11}$$

$$\Delta_2(\Gamma) = \Delta_2 = m_{12} - \Gamma_{11}.$$

The action of the Weyl reflection r_1 on the representation labels is

$$r_1 * [m_{12}, m_{22}] = [m_{22}-1, m_{12}+1].$$

We verify that the vertex operator (3.9) trivially obeys the equivariance condition (2.6):

$$s_{12}^{m_{12}+\Delta_1-m_{22}-\Delta_2+1} \left\langle \begin{matrix} & \Gamma_{11} & \\ m_{12}^{(t)} & & 0 \\ & m_{12}^{(t)} & \end{matrix} \right\rangle = \left\langle \begin{matrix} & m_{12}^{(t)} - \Gamma_{11} & \\ m_{12}^{(t)} & & 0 \\ & m_{12}^{(t)} & \end{matrix} \right\rangle s_{12}^{m_{12}-m_{22}+1}.$$

Correspondingly, it has

$$\left[\frac{\Delta_1!\Delta_2!}{(\Delta_1+\Delta_2)!} \frac{(p_{12}-p_{22}-1)!}{(p_{12}-p_{22}-\Delta_2-1)!} \frac{(p_{12}+\Delta_1-p_{22})!}{(p_{12}+\Delta_1-p_{22}-\Delta_2)!} \right]^{1/2} \tag{3.10}$$

as its su(2) reduced matrix element when acting on the irrep $[m_{12}m_{22}]$ as easily computed from equation (3.8) and elementary su(2) recoupling coefficients (see e.g. Le Blanc and Hecht 1987, appendix).

The lower weight components of tensor (3.9) are easily obtained through the usual lowering procedure and one has

$$\left\langle \begin{matrix} & \Gamma_{11} & \\ m_{12} & & 0 \\ & m_{11} & \end{matrix} \right\rangle = [(m_{12}-m_{11})!]^{-1/2} K^{-1} \begin{pmatrix} m_{12} & & 0 \\ & m_{11} & \end{pmatrix}$$
$$\times [\sigma(E_{21})]^{(m_{12}-m_{11})} \left\{ \left\langle \begin{matrix} & \Gamma_{11} & \\ m_{12} & & 0 \\ & m_{12} & \end{matrix} \right\rangle \right\} \tag{3.11}$$

where

$$A^{(k)}\{X\} = \underbrace{[A,[A,\ldots,[A,X]\ldots]]}_{k \text{ commutators}}.$$

It is relatively straightforward to prove by induction that

$$\left\langle \begin{matrix} m_{12} & \begin{matrix}\Gamma_{11}\\ \\ m_{11}\end{matrix} & 0 \end{matrix} \right\rangle = (-1)^{m_{12}-\Gamma_{11}}[(m_{12}-m_{11})!]^{-1/2}K^{-1}\begin{pmatrix} m_{12} & & 0 \\ & m_{11} & \end{pmatrix}$$
$$\times \sum_{j=0}^{k=m_{12}-m_{11}} (-1)^j \binom{k}{j} (\Gamma_{11}-k+j+1)_{k-j}(m_{12}-\Gamma_{11}-j+1)_j z_{12}^{k-j}$$
$$\times \mathrm{e}^{\mathrm{i}(\Gamma_{11}q_1+jq_2)}(\mathrm{e}^{\mathrm{i}q_2}\partial_{12})^{m_{12}-\Gamma_{11}-j}(p_1-p_2-z_{12}\partial_{12}-j+1)_j \qquad (3.12)$$

in terms of the rising factorial

$$(x)_j = (x)(x+1)\ldots(x+j-1) \qquad (x)_0 = 1.$$

Matrix elements between states belonging to the basis (3.2) are then easily computed.

With the help of the little compendium

$$\begin{aligned}
k_1 &= \Gamma_{11} \\
k_2 &= m_{12}^{(i)} - m_{22}^{(i)} + \Gamma_{11} - m_{12}^{(t)} \\
k_3 &= m_{12}^{(t)} - \Gamma_{11} \\
s_1 &= m_{11}^{(i)} - m_{22}^{(i)} \\
s_2 &= m_{11}^{(t)} \\
s_3 &= m_{12}^{(i)} - m_{11}^{(i)} + \Gamma_{11} - m_{11}^{(t)} \\
d_1 &= m_{12}^{(i)} - m_{11}^{(i)} \\
d_2 &= m_{12}^{(t)} - m_{11}^{(t)} \\
d_3 &= \Gamma_{11} - m_{22}^{(i)} + m_{11}^{(i)} - m_{12}^{(t)} + m_{11}^{(t)}
\end{aligned} \qquad (3.13)$$

one finds for the *normalized* vertex operator

$$\left\langle \begin{matrix} m_{12}^{(f)} & & m_{22}^{(f)} \\ & m_{11}^{(f)} & \end{matrix} \middle| \begin{bmatrix} m_{12}^{(t)} & \begin{matrix}\Gamma_{11}\\ \\ m_{11}^{(t)}\end{matrix} & 0 \end{bmatrix} \middle| \begin{matrix} m_{12}^{(i)} & & m_{22}^{(i)} \\ & m_{11}^{(i)} & \end{matrix} \right\rangle$$
$$= (-1)^{d_3-k_1}[m_{12}^{(f)} - m_{22}^{(f)} + 1]^{1/2} \begin{bmatrix} s_1 & s_2 & s_3 \\ d_1 & d_2 & d_3 \\ k_1 & k_2 & k_3 \end{bmatrix} \qquad (3.14)$$

where the last square array is the well known Regge symbol (Regge 1958) (a magic square with the sum for all rows and columns given by J) here computed to be given by

$$\begin{bmatrix} s_1 & s_2 & s_3 \\ d_1 & d_2 & d_3 \\ k_1 & k_2 & k_3 \end{bmatrix} = (-1)^{d_3-s_2}\left[\frac{1}{(J+1)!}\frac{d_1!d_3!}{d_2!}\frac{s_2!k_2!}{s_1!k_1!s_3!k_3!}\right]^{1/2} P_{d_2}(s_1,k_3;s_3,k_1)$$

$$= (-1)^{s_1-d_2}\left[\frac{1}{(J+1)!}\frac{k_1!k_2!}{k_3!}\frac{s_3!d_3!}{s_1!d_1!s_2!d_2!}\right]^{1/2} P_{k_3}(s_1,d_2;s_2,d_1)$$

where P is a polynomial given by

$$P_{k_3}(s_1,d_2;s_2,d_1) = \sum_{j=0}^{k_3}(-1)^j\binom{k_3}{j}(-s_1)_{k_3-j}(-d_2)_{k_3-j}(-s_2)_j(-d_1)_j .$$

4. Vertex operators for u(3)

4.1. Left, right, unitary action and screening operators for u(3)

The realization σ_Λ of the finite-dimensional simple Lie algebra su(3) in terms of linear differential operators acting on the space of holomorphic sections of a line bundle over the flag manifold $H\backslash\mathrm{SU}(3) \sim \mathrm{B}_-\backslash\mathrm{SL}(3)$ determined by the character $\chi_\Lambda : \mathrm{B}_- \to \mathbf{C}^*$ is obtained through the equality

$$\sigma_\Lambda(X)\langle\Lambda \mid \exp(z) \mid \psi\rangle = \langle\Lambda \mid \exp(z)X \mid \psi\rangle \qquad X \in \mathrm{su}(3) \tag{4.1}$$

where the flag manifold is here parametrized by

$$\exp(z) = \exp(z_{12}E_{12} + z_{23}E_{23} + z_{13}E_{13}).$$

We find

$$\begin{aligned}
&\sigma_\Lambda(E_{13}) = \partial_{13} \qquad \sigma_\Lambda(E_{12}) = \partial_{12} - \tfrac{1}{2}z_{23}\partial_{13} \qquad \sigma_\Lambda(E_{23}) = \partial_{23} + \tfrac{1}{2}z_{12}\partial_{13} \\
&\sigma_\Lambda(E_{11} - E_{22}) = \Lambda_1 - 2z_{12}\partial_{12} + z_{23}\partial_{23} - z_{13}\partial_{13} \\
&\sigma_\Lambda(E_{22} - E_{33}) = \Lambda_2 + z_{12}\partial_{12} - 2z_{23}\partial_{23} - z_{13}\partial_{13} \\
&\sigma_\Lambda(E_{21}) = \Lambda_1 z_{12} - z_{13}\partial_{23} + z_{12}\left(\tfrac{1}{2}z_{23}\partial_{23} - z_{12}\partial_{12} - \tfrac{1}{2}z_{13}\partial_{13} - \tfrac{1}{4}z_{12}z_{23}\partial_{13}\right) \\
&\sigma_\Lambda(E_{32}) = \Lambda_2 z_{23} + z_{13}\partial_{12} + z_{23}\left(\tfrac{1}{2}z_{12}\partial_{12} - z_{23}\partial_{23} - \tfrac{1}{2}z_{13}\partial_{13} + \tfrac{1}{4}z_{12}z_{23}\partial_{13}\right) \\
&\sigma_\Lambda(E_{31}) = (\Lambda_1 + \Lambda_2)z_{13} + \tfrac{1}{2}z_{12}z_{23}(\Lambda_1 - \Lambda_2) \\
&\qquad - z_{13}\left(z_{12}\partial_{12} + z_{23}\partial_{23} + z_{13}\partial_{13}\right) + \tfrac{1}{2}z_{12}z_{23}^2\partial_{23} \\
&\qquad - \tfrac{1}{4}z_{12}^2z_{23}^2\partial_{13} - \tfrac{1}{2}z_{12}^2z_{23}\partial_{12}.
\end{aligned} \tag{4.2}$$

This representation for su(3) can be untangled by performing the following non-linear change of variables:

$$z'_{12} = z_{12}$$
$$z'_{23} = z_{23}$$
$$z'_{13} = z_{13} - \tfrac{1}{2} z_{12} z_{23}.$$

In terms of the new variables, and adjoining the three pairs of momentum–position operators (p_i, q_i) with

$$[p_i, q_j] = -i\delta_{ij} \qquad 1 \leq i, j \leq 3$$

the expansion for the non-simple algebra u(3) can be written (we ignore the primes on the new variables)

$$\begin{aligned}
\sigma(E_{13}) &= \partial_{13} \\
\sigma(E_{23}) &= \partial_{23} \\
\sigma(E_{12}) &= \mathcal{E}_{12} - z_{23}\partial_{13} \\
\sigma(E_{11}) &= \mathcal{E}_{11} - z_{13}\partial_{13} \\
\sigma(E_{22}) &= \mathcal{E}_{22} - z_{23}\partial_{23} \\
\sigma(E_{21}) &= \mathcal{E}_{21} - z_{13}\partial_{23} \\
\sigma(E_{33}) &= \mathcal{E}_{33} + \sum_{i=1}^{2} z_{i3}\partial_{i3} \\
\sigma(E_{31}) &= \sum_{i=1}^{2} z_{i3}\mathcal{E}_{i1} - z_{13}\mathcal{E}_{33} - z_{13}\sum_{i=1}^{2} z_{i3}\partial_{i3} = [\Xi, z_{13}] \\
\sigma(E_{32}) &= \sum_{i=1}^{2} z_{i3}\mathcal{E}_{i2} - z_{23}\mathcal{E}_{33} - z_{23}\sum_{i=1}^{2} z_{i3}\partial_{i3} = [\Xi, z_{23}]
\end{aligned} \tag{4.3}$$

where

(i) the operators

$$\begin{aligned}
\mathcal{E}_{12} &= \partial_{12} \\
\mathcal{E}_{11} &= p_1 - z_{12}\partial_{12} \\
\mathcal{E}_{22} &= p_2 + z_{12}\partial_{12} \\
\mathcal{E}_{21} &= z_{12}(p_1 - p_2) - z_{12}^2\partial_{12}
\end{aligned} \tag{4.4}$$

generate an *intrinsic* u(2) algebra isomorphic to the one given by (3.2),

(ii) the operator

$$\mathcal{E}_{33} = p_3$$

generates an *intrinsic* u(1) and

(iii) Ξ is the u(2) scalar (Hecht *et al* 1987)

$$\Xi = (z_{i3}\partial_{j3})(\mathcal{E}_{ij} - \delta_{ij}\mathcal{E}_{33}) - \tfrac{1}{2}(z_{i3}\partial_{j3})(z_{j3}\partial_{i3}) + (z_{k3}\partial_{k3})$$

where repeated indices are summed from 1 to 2.

Expansion (4.3) can be recognized as the *vector coherent state* expansion for u(3) (Hecht *et al* 1987). In this framework, the irreducible space is now carried by holomorphic sections of a *vector* bundle with the fibre carrying an irrep of the *intrinsic* u(2)$\oplus$u(1), labelled by $[m_{13}m_{23}]\otimes[m_{33}]$, while the base manifold is now parametrized by the pair of coordinates (z_{13}, z_{23}) transforming as an antispinor $[0,-1]$ under the u(2) algebra generated by $\sigma(E_{ij})$, $1 \le i,j \le 2$. (This base manifold is in fact a Kähler manifold isomorphic to $(\mathrm{SU}(2)\times\mathrm{U}(1))\backslash\mathrm{SU}(3)$ (Bordemann *et al* 1986).)

It has been shown by Hecht *et al* (1987) how to unitarize the action of the linear differential operators (4.3) on the (vector-coupled) Fock space of polynomials in the antispinor variables. For u(3), as generated by (4.3), the action is unitary on the u(2)-coupled basis

$$\left|\begin{matrix} m_{13} & & m_{23} & & m_{33} \\ & m_{12} & & m_{22} & \\ & & m_{11} & & \end{matrix}\right\rangle = (-1)^{\phi([m_{13},m_{23}])-\phi([0,-w])-\phi([m_{12},m_{22}])}$$
$$\times K\begin{pmatrix} m_{13} & & m_{23} & & m_{33} \\ & m_{12} & & m_{22} & \end{pmatrix} \mathrm{e}^{\mathrm{i}m_{33}q_3}$$
$$\times \left[\mathcal{Z}^{[0,-w]}_{*}(z)\left|\begin{matrix} m_{13} & & m_{23} \\ & * & \end{matrix}\right\rangle\right]^{[m_{12}m_{22}]}_{m_{11}} \tag{4.5}$$

where the coupling is from right to left and

(i) where

$$w = m_{13} + m_{23} - m_{12} - m_{22}$$

is the eigenvalue of the operator $\sigma(E_{33}) - \mathcal{E}_{33} = \sum_{i=1}^{2} z_{i3}\partial_{i3}$ (the antispinor z-number operator),

(ii) where $\mathcal{Z}$ is a polynomial in the antispinor variables (z_{13}, z_{23}) with highest weight component

$$\mathcal{Z}^{[0,-w]}_{0}(z) = (-1)^{w}\frac{z_{23}^{w}}{\sqrt{w!}}$$

(iii) where the vector coherent state unitarizing factor is given by

$$K\begin{pmatrix} [m_n\phantom{_{-1}}] \\ [m_{n-1}] \end{pmatrix} = \left[\prod_{i=1}^{n-1}\frac{(p_{in} - p_{nn} - 1)!}{(p_{i,n-1} - p_{nn})!}\right]^{1/2}$$

(set $n = 3$ here),

(iv) where the su(2) phase factor is given by

$$\phi([m_2]) = (\rho_{su(2)}, m_2) = \tfrac{1}{2}(m_{12} - m_{22})$$

(v) and where the intrinsic vectors (the vectorial structure of the fibre)

$$\left| \begin{matrix} m_{13} & & m_{23} \\ & m_{11} & \end{matrix} \right\rangle$$

have been constructed in section 3, equation (3.3).

The highest weight state for this representation is given by

$$|hw\rangle = \mathrm{e}^{\mathrm{i}(m_{13}q_1 + m_{23}q_2 + m_{33}q_3)}|0\rangle.$$

The left action for the raising algebra $\mathbf{n}_+$ is determined through the equality

$$\rho(X)\langle hw \mid \exp(z) \mid \psi\rangle = \langle hw \mid (-X)\exp(z) \mid \psi\rangle \qquad X \in \mathbf{n}_+.$$

The minus sign in this equation results in the vertex operators having phases consistent with a generalized Condon–Shortley phase convention (Biedenharn and Louck 1968). One finds

$$\begin{aligned} \rho(E_{12}) &= -\partial_{12} \\ \rho(E_{23}) &= -(\partial_{23} - z_{12}\partial_{13}) \\ \rho(E_{13}) &= -\partial_{13}. \end{aligned} \tag{4.6}$$

The three screening charges (two simple and one derived) for u(3) are therefore given by

$$\begin{aligned} s_{12} &= \mathrm{e}^{-\mathrm{i}(q_1 - q_2)}\rho(E_{12}) \\ s_{23} &= \mathrm{e}^{-\mathrm{i}(q_2 - q_3)}\rho(E_{23}) \\ s_{13} &= \mathrm{e}^{-\mathrm{i}(q_1 - q_3)}\rho(E_{13}) = [s_{12}, s_{23}]. \end{aligned} \tag{4.7}$$

Under the algebra (4.3), the screening charges have null weight and map the irrep $[m_3]$ as follows

$$\begin{aligned} s_{12} : [m_{13}m_{23}m_{33}] &\rightarrow [m_{13} - 1, m_{23} + 1, m_{33}] \\ s_{23} : [m_{13}m_{23}m_{33}] &\rightarrow [m_{13}, m_{23} - 1, m_{33} + 1] \\ s_{13} : [m_{13}m_{23}m_{33}] &\rightarrow [m_{13} - 1, m_{23}, m_{33} + 1]. \end{aligned}$$

The Weyl reflections effect the changes

$$\begin{aligned} r_1 * [m_{13}m_{23}m_{33}] &= [m_{23} - 1, m_{13} + 1, m_{33}] \\ r_2 * [m_{13}m_{23}m_{33}] &= [m_{13}, m_{33} - 1, m_{23} + 1] \end{aligned} \tag{4.8}$$

on the representation labels which are equivalent to the permutations

$$\begin{aligned} p_{13} &\leftrightarrow p_{23} \\ p_{23} &\leftrightarrow p_{33} \end{aligned}$$

in terms of the partial hooks.

4.2. The elementary tensors and their fusion rules

The fundamental u(3) tensor [100] has three possible shifts: $\Delta(\Gamma) = (1,0,0)$, $(0,1,0)$ and $(0,0,1)$. Under Weyl reflections, the final irreps read as follows:

$[m_3] + [\Delta(\Gamma)]$	$r_1 * ([m_3] + [\Delta(\Gamma)])$	$r_2 * ([m_3] + [\Delta(\Gamma)])$
$[m_{13}+1, m_{23}, m_{33}]$	$[m_{23}-1, m_{13}+2, m_{33}]$	$[m_{13}+1, m_{33}-1, m_{23}+1]$
$[m_{13}, m_{23}+1, m_{33}]$	$[m_{23}, m_{13}+1, m_{33}]$	$[m_{13}, m_{33}-1, m_{23}+2]$
$[m_{13}, m_{23}, m_{33}+1]$	$[m_{23}-1, m_{13}+1, m_{33}+1]$	$[m_{13}, m_{33}, m_{23}+1]$

Using the equivariance condition (2.6) and the table, we now seek to construct the corresponding vertex operators. For example, we must verify that

$$s_{23}^{m_{23}-m_{33}+2} \left\langle \begin{array}{ccccc} & & 0 & & \\ & 1 & & 0 & \\ 1 & & 0 & & 0 \\ & & hw & & \end{array} \right\rangle_{\mathrm{un}} = \left\langle \begin{array}{ccccc} & & 0 & & \\ & 0 & & 0 & \\ 1 & & 0 & & 0 \\ & & hw & & \end{array} \right\rangle_{\mathrm{un}} s_{23}^{m_{23}-m_{33}+1} .$$

We find for the three fundamental (unnormalized) vertex operators

$$\begin{aligned} \left\langle \begin{array}{ccccc} & & 1 & & \\ & 1 & & 0 & \\ 1 & & 0 & & 0 \\ & & hw & & \end{array} \right\rangle_{\mathrm{un}} &= \left\langle \begin{array}{ccc} & 1 & \\ 1 & & 0 \\ & hw & \end{array} \right\rangle_{\mathrm{un}} = \mathrm{e}^{\mathrm{i}q_1} \\ \left\langle \begin{array}{ccccc} & & 0 & & \\ & 1 & & 0 & \\ 1 & & 0 & & 0 \\ & & hw & & \end{array} \right\rangle_{\mathrm{un}} &= \left\langle \begin{array}{ccc} & 0 & \\ 1 & & 0 \\ & hw & \end{array} \right\rangle_{\mathrm{un}} = \mathrm{e}^{\mathrm{i}q_1} s_{12} \\ \left\langle \begin{array}{ccccc} & & 0 & & \\ & 0 & & 0 & \\ 1 & & 0 & & 0 \\ & & hw & & \end{array} \right\rangle_{\mathrm{un}} &= \mathrm{e}^{\mathrm{i}q_1} [s_{12}s_{23} + s_{13}(p_2 - p_3)] \\ &= \mathrm{e}^{\mathrm{i}q_3} [\partial_{13}, \Omega]. \end{aligned} \tag{4.9}$$

Reduced vertex operators for the first two tensors in (4.9) are easily obtained and correspond to the case of maximally tied upper patterns (Le Blanc and Biedenharn 1989, section 4.7). Of particular interest, however, is the third vertex operator given previously. This operator realization has a non-maximally tied upper pattern and has not been obtained previously in such an explicit form. In tensorial notation, the u(2) spinor of this tensor is given by the commutator

$$\mathrm{e}^{\mathrm{i}q_3} [\partial^{[10]}, \Omega] \tag{4.10a}$$

where Ω is the u(2) scalar

$$\Omega = z_{i3}\partial_{j3}\mathcal{E}_{ij} - (\mathcal{E}_{11} + \mathcal{E}_{22} - \mathcal{E}_{33} + 1)(z_{i3}\partial_{i3}) \tag{4.10b}$$

with repeated indices running from 1 to 2. The u(2) singlet, obtained through standard lowering techniques, is given by

$$-\,\mathrm{e}^{iq_3}\left((\mathcal{E}_{12}-z_{23}\partial_{13})(\mathcal{E}_{21}-z_{13}\partial_{23})-(\mathcal{E}_{22}-z_{23}\partial_{23}-\mathcal{E}_{33}+1)(\mathcal{E}_{11}-z_{13}\partial_{13}-\mathcal{E}_{33})\right) \tag{4.11a}$$

and has matrix element

$$\left[\prod_{i=1}^{2}(p_{i3}-p_{33}-1)\right]\frac{K^2\begin{pmatrix} m_{13} & & m_{23} & & m_{33}+1 \\ & m_{12} & & m_{22} & \end{pmatrix}}{K^2\begin{pmatrix} m_{13} & & m_{23} & & m_{33} \\ & m_{12} & & m_{22} & \end{pmatrix}} \tag{4.11b}$$

when acting on the basis state (4.5). Computation of the (unnormalized) reduced vertex operators is greatly simplified by these observations. For example, we obtain

$$\begin{aligned}
&\left\langle \begin{matrix} m_{13} & & m_{23} & & m_{33}+1 \\ & m_{12}+1 & & m_{22} & \end{matrix} \right\| \begin{bmatrix} & & 0 & & \\ & 0 & & 0 & \\ 1 & & 0 & & 0 \\ & 1 & & 0 & \\ & & 1 & & \end{bmatrix}_{\mathrm{un}} \left\| \begin{matrix} m_{13} & & m_{23} & & m_{33} \\ & m_{12} & & m_{22} & \end{matrix} \right\rangle \\
&\quad = (p_{33}-p_{22})\begin{bmatrix} [m_{13}m_{23}] & [0-w] & [m_{12}m_{22}] \\ [00] & [10] & [10] \\ [m_{13}m_{23}] & [0-(w-1)] & [m_{12}+1,m_{22}] \end{bmatrix} \\
&\quad\quad \times \langle [0,-(w-1)] \| \mathcal{Z}^{[10]} \| [0,-w] \rangle \frac{K\begin{pmatrix} [m_{13}m_{23}m_{33}] \\ [m_{12},m_{22}] \end{pmatrix}}{K\begin{pmatrix} [m_{13},m_{23},m_{33}+1] \\ [m_{12}+1,m_{22}] \end{pmatrix}}
\end{aligned}$$

where $[\ldots]$ is a unitary u(2) 9-j recoupling coefficient, $\langle [0,-(w-1)] \| \mathcal{Z}^{[10]} \| [0,-w] \rangle$ is an antispinor reduced matrix element, and the factor $(p_{33}-p_{22})$ amounts to a difference of eigenvalues of the u(2) invariant operator Ω. This matrix element and all other matrix elements pertaining to the elementary vertex operators with $\Delta_3(\Gamma)=1$ have been listed in table 1.

It is straightforward to normalize the third fundamental reduced vertex operator by dividing it by its su(3) reduced matrix element

$$[(p_{33}-p_{13}+1)(p_{33}-p_{13})(p_{33}-p_{23}+1)(p_{33}-p_{23})]^{1/2}. \tag{4.12}$$

One then retrieves the results and phase convention of the pattern calculus of Biedenharn and Louck (1968) for this fundamental projective operator.

A second elementary u(3) vertex operator [110] exists with three possible shifts $\Delta(\Gamma)=(1,1,0)$, $(1,0,1)$ and $(0,1,1)$. Under Weyl reflections, the final irreps are:

$[m_3]+[\Delta(\Gamma)]$	$r_1*([m_3]+[\Delta(\Gamma)])$	$r_2*([m_3]+[\Delta(\Gamma)])$
$[m_{13}+1,m_{23}+1,m_{33}]$	$[m_{23},m_{13}+2,m_{33}]$	$[m_{13}+1,m_{33}-1,m_{23}+2]$
$[m_{13}+1,m_{23},m_{33}+1]$	$[m_{23}-1,m_{13}+2,m_{33}+1]$	$[m_{13}+1,m_{33},m_{23}+1]$
$[m_{13},m_{23}+1,m_{33}+1]$	$[m_{23},m_{13}+1,m_{33}+1]$	$[m_{13},m_{33},m_{23}+2]$

Using this table and equation (2.6), we find the following (unnormalized) vertex operators

$$\left\langle \begin{array}{ccccc} & & 1 & & \\ & 1 & & 1 & \\ 1 & & 1 & & 0 \\ & & hw & & \end{array} \right\rangle_{\rm un} = \left\langle \begin{array}{ccc} & 1 & \\ 1 & & 1 \\ & hw & \end{array} \right\rangle_{\rm un} = \mathrm{e}^{\mathrm{i}(q_1+q_2)}$$

$$\left\langle \begin{array}{ccccc} & & 1 & & \\ & 1 & & 0 & \\ 1 & & 1 & & 0 \\ & & hw & & \end{array} \right\rangle_{\rm un} = \mathrm{e}^{\mathrm{i}(q_1+q_2)} s_{23}$$

$$= -\mathrm{e}^{\mathrm{i}q_3}\sqrt{2}\left[\partial^{[10]} \times \left\langle \begin{array}{ccc} & 1 & \\ 1 & & 0 \\ & * & \end{array} \right\rangle\right]^{[11]}_{hw}$$

$$\left\langle \begin{array}{ccccc} & & 0 & & \\ & 1 & & 0 & \\ 1 & & 1 & & 0 \\ & & hw & & \end{array} \right\rangle_{\rm un} = \mathrm{e}^{\mathrm{i}(q_1+q_2)}[s_{12}s_{23} - s_{13}(p_1 - p_2 + 1)]$$

$$= -\mathrm{e}^{\mathrm{i}q_3}\sqrt{2}\left[\partial^{[10]} \times \left\langle \begin{array}{ccc} & 0 & \\ 1 & & 0 \\ & * & \end{array} \right\rangle\right]^{[11]}_{hw}$$

which we have also given in tensorial notation involving the u(2) vertex tensor operators studied in section 3. The [110] vertex operators with $\Delta_3(\Gamma) = 1$ given previously have not been obtained previously in this explicit form. Matrix elements for these operators have been computed and listed in table 1.

4.3. Structure of the elementary vertex operators

A close look at table 1 shows that there is an underlying structure to these results which will enable us to recast all of table 1 into a single formula (given later). But first let us make a few additional notational comments (Biedenharn and Louck 1968).

Elementary tensors for u(n) are antisymmetric tensors labelled by

$$[\dot{1}_k\dot{0}_{n-k}] \equiv [\underbrace{11\ldots1}_{k\text{ times}}\ \underbrace{00\ldots0}_{(n-k)\text{ times}}\ ].$$

The shifts $\Delta(\Gamma)$ and $\Delta(\gamma)$ for these operators are permutations of these partitions. These shifts can be succinctly given by

$$\Delta(\Gamma) = (i_1 i_2 \ldots i_k)$$

and

$$\Delta(\gamma) = (j_1 j_2 \cdots j_k)$$

where, for example, $i_1 < i_2 < \ldots i_k$ denotes the k places in the $\Delta(\Gamma)$ array where the 1s occurs (with 0s in all other places). For u(3), $k = 1$ or 2 only.

Table 1(*a*). Vertex operator [100] with shifts [101]. Normalizing factor: $[(p_{33}-p_{13}+1)(p_{33}-p_{13})(p_{33}-p_{23}+1)(p_{33}-p_{23})]^{1/2}$.

$$\left\langle \begin{array}{ccccc} m_{13} & & m_{23} & & m_{33}+1 \\ & m_{12}+1 & & m_{22} & \end{array} \right| \left[\begin{array}{ccccc} & & 0 & & \\ & 0 & & 0 & \\ 1 & & 0 & & 0 \\ & 1 & & 0 & \\ & & 1 & & \end{array} \right]_{\mathrm{un}} \left| \begin{array}{ccccc} m_{13} & & m_{23} & & m_{33} \\ & m_{12} & & m_{22} & \end{array} \right\rangle$$

$$= (p_{33}-p_{22}) \begin{bmatrix} [m_{13}m_{23}] & [0,-w] & [m_{12}m_{22}] \\ [00] & [10] & [10] \\ [m_{13}m_{23}] & [0-(w-1)] & [m_{12}+1,m_{22}] \end{bmatrix} \langle [0,-(w-1)] ||| \partial^{[10]} ||[0,-w]\rangle \frac{K\begin{pmatrix} [m_{13}m_{23}m_{33}] \\ [m_{12},m_{22}] \end{pmatrix}}{K\begin{pmatrix} [m_{13},m_{23},m_{33}+1] \\ [m_{12}+1,m_{22}] \end{pmatrix}}$$

$$\left\langle \begin{array}{ccccc} m_{13} & & m_{23} & & m_{33}+1 \\ & m_{12} & & m_{22}+1 & \end{array} \right| \left[\begin{array}{ccccc} & & 0 & & \\ & 0 & & 0 & \\ 1 & & 0 & & 0 \\ & 1 & & 0 & \\ & & 0 & & \end{array} \right]_{\mathrm{un}} \left| \begin{array}{ccccc} m_{13} & & m_{23} & & m_{33} \\ & m_{12} & & m_{22} & \end{array} \right\rangle$$

$$= (-1)(p_{33}-p_{12}) \begin{bmatrix} [m_{13}m_{23}] & [0,-w] & [m_{12}m_{22}] \\ [00] & [10] & [10] \\ [m_{13}m_{23}] & [0-(w-1)] & [m_{12},m_{22}+1] \end{bmatrix} \langle [0,-(w-1)] ||| \partial^{[10]} ||[0,-w]\rangle \frac{K\begin{pmatrix} [m_{13}m_{23}m_{33}] \\ [m_{12},m_{22}] \end{pmatrix}}{K\begin{pmatrix} [m_{13},m_{23},m_{33}+1] \\ [m_{12},m_{22}+1] \end{pmatrix}}$$

$$\left\langle \begin{array}{ccccc} m_{13} & & m_{23} & & m_{33}+1 \\ & m_{12} & & m_{22} & \end{array} \right| \left[\begin{array}{ccccc} & & 0 & & \\ & 0 & & 0 & \\ 1 & & 0 & & 0 \\ & 0 & & 0 & \\ & & 0 & & \end{array} \right]_{\mathrm{un}} \left| \begin{array}{ccccc} m_{13} & & m_{23} & & m_{33} \\ & m_{12} & & m_{22} & \end{array} \right\rangle$$

$$= (p_{33}-p_{12})(p_{33}-p_{22}) \begin{bmatrix} [m_{13}m_{23}] & [0,-w] & [m_{12}m_{22}] \\ [00] & [00] & [00] \\ [m_{13}m_{23}] & [0,-w] & [m_{12},m_{22}] \end{bmatrix} \frac{K\begin{pmatrix} [m_{13}m_{23}m_{33}] \\ [m_{12},m_{22}] \end{pmatrix}}{K\begin{pmatrix} [m_{13},m_{23},m_{33}+1] \\ [m_{12},m_{22}] \end{pmatrix}}$$

Table 1(b). Vertex operator [110] with shifts [101]. Normalizing factor: $[(p_{33}-p_{23})(p_{33}-p_{23}+1)]^{1/2}$.

$$\left\langle \begin{array}{ccccc} m_{13}+1 & & m_{23} & & m_{33}+1 \\ & m_{12}+1 & & m_{22}+1 & \end{array} \right| \left[\begin{array}{ccccc} & & 1 & & \\ & 1 & & 0 & \\ 1 & & 1 & & 0 \\ & 1 & & 1 & \\ & & 1 & & \end{array} \right]_{\mathrm{un}} \left| \begin{array}{ccccc} m_{13} & & m_{23} & & m_{33} \\ & m_{12} & & m_{22} & \end{array} \right\rangle$$

$$= \sqrt{2} \begin{bmatrix} [m_{13}m_{23}] & [0,-w] & [m_{12}m_{22}] \\ [10] & [10] & [11] \\ [m_{13}+1, m_{23}] & [0-(w-1)] & [m_{12}+1, m_{22}+1] \end{bmatrix} \langle [0,-(w-1)] \| \partial^{[10]} \| [0,-w] \rangle \frac{K\begin{pmatrix} [m_{13}m_{23}m_{33}] \\ [m_{12}, m_{22}] \end{pmatrix}}{K\begin{pmatrix} [m_{13}+1, m_{23}, m_{33}+1] \\ [m_{12}+1, m_{22}+1] \end{pmatrix}}$$

$$\left\langle \begin{array}{ccccc} m_{13}+1 & & m_{23} & & m_{33}+1 \\ & m_{12}+1 & & m_{22} & \end{array} \right| \left[\begin{array}{ccccc} & & 1 & & \\ & 1 & & 0 & \\ 1 & & 1 & & 0 \\ & 1 & & 0 & \\ & & 1 & & \end{array} \right]_{\mathrm{un}} \left| \begin{array}{ccccc} m_{13} & & m_{23} & & m_{33} \\ & m_{12} & & m_{22} & \end{array} \right\rangle$$

$$= -(p_{33}-p_{22}) \begin{bmatrix} [m_{13}m_{23}] & [0,-w] & [m_{12}m_{22}] \\ [10] & [00] & [10] \\ [m_{13}+1, m_{23}] & [0,-w] & [m_{12}+1, m_{22}] \end{bmatrix} \frac{K\begin{pmatrix} [m_{13}m_{23}m_{33}] \\ [m_{12}, m_{22}] \end{pmatrix}}{K\begin{pmatrix} [m_{13}+1, m_{23}, m_{33}+1] \\ [m_{12}+1, m_{22}] \end{pmatrix}}$$

$$\left\langle \begin{array}{ccccc} m_{13}+1 & & m_{23} & & m_{33}+1 \\ & m_{12} & & m_{22}+1 & \end{array} \right| \left[\begin{array}{ccccc} & & 1 & & \\ & 1 & & 0 & \\ 1 & & 1 & & 0 \\ & 1 & & 0 & \\ & & 0 & & \end{array} \right]_{\mathrm{un}} \left| \begin{array}{ccccc} m_{13} & & m_{23} & & m_{33} \\ & m_{12} & & m_{22} & \end{array} \right\rangle$$

$$= (p_{33}-p_{12}) \begin{bmatrix} [m_{13}m_{23}] & [0,-w] & [m_{12}m_{22}] \\ [10] & [00] & [10] \\ [m_{13}+1, m_{23}] & [0,-w] & [m_{12}, m_{22}+1] \end{bmatrix} \frac{K\begin{pmatrix} [m_{13}m_{23}m_{33}] \\ [m_{12}, m_{22}] \end{pmatrix}}{K\begin{pmatrix} [m_{13}+1, m_{23}, m_{33}+1] \\ [m_{12}, m_{22}+1] \end{pmatrix}}$$

Table 1(c). Vertex operator [110] with shifts [011]. Normalizing factor: $[(p_{33}-p_{13}+1)(p_{33}-p_{13})(p_{23}-p_{13}+1)(p_{23}-p_{13})]^{1/2}$.

$$\left\langle \begin{matrix} m_{13} & & m_{23}+1 & & m_{33}+1 \\ & m_{12}+1 & & m_{22}+1 & \end{matrix} \right\| \left[\begin{matrix} & & 0 & & \\ & 1 & & 0 & \\ 1 & & 1 & & 0 \\ & 1 & & 1 & \\ & & 1 & & \end{matrix} \right]_{\mathrm{un}} \left| \begin{matrix} m_{13} & & m_{23} & & m_{33} \\ & m_{12} & & m_{22} & \end{matrix} \right\rangle$$

$$= -\sqrt{2}[(p_{23}-p_{13})(p_{23}-p_{13}+1)]^{1/2}$$

$$\times \begin{bmatrix} [m_{13}m_{23}] & [0,-w] & [m_{12}m_{22}] \\ [10] & [10] & [11] \\ [m_{13}, m_{23}+1] & [0-(w-1)] & [m_{12}+1, m_{22}+1] \end{bmatrix} \langle [0,-(w-1)] \| \partial^{[10]} \| [0,-w] \rangle \frac{K\begin{pmatrix} [m_{13}m_{23}m_{33}] \\ [m_{12}, m_{22}] \end{pmatrix}}{K\begin{pmatrix} [m_{13}, m_{23}+1, m_{33}+1] \\ [m_{12}+1, m_{22}+1] \end{pmatrix}}$$

$$\left\langle \begin{matrix} m_{13} & & m_{23}+1 & & m_{33}+1 \\ & m_{12}+1 & & m_{22} & \end{matrix} \right\| \left[\begin{matrix} & & 0 & & \\ & 1 & & 0 & \\ 1 & & 1 & & 0 \\ & 1 & & 0 & \\ & & 1 & & \end{matrix} \right]_{\mathrm{un}} \left| \begin{matrix} m_{13} & & m_{23} & & m_{33} \\ & m_{12} & & m_{22} & \end{matrix} \right\rangle$$

$$= (p_{33}-p_{22})[(p_{23}-p_{13})(p_{23}-p_{13}+1)]^{1/2}$$

$$\times \begin{bmatrix} [m_{13}m_{23}] & [0,-w] & [m_{12}m_{22}] \\ [10] & [00] & [10] \\ [m_{13}, m_{23}+1] & [0,-w] & [m_{12}+1, m_{22}] \end{bmatrix} \frac{K\begin{pmatrix} [m_{13}m_{23}m_{33}] \\ [m_{12}, m_{22}] \end{pmatrix}}{K\begin{pmatrix} [m_{13}, m_{23}+1, m_{33}+1] \\ [m_{12}+1, m_{22}] \end{pmatrix}}$$

$$\left\langle \begin{matrix} m_{13} & & m_{23}+1 & & m_{33}+1 \\ & m_{12} & & m_{22}+1 & \end{matrix} \right\| \left[\begin{matrix} & & 0 & & \\ & 1 & & 0 & \\ 1 & & 1 & & 0 \\ & 1 & & 0 & \\ & & 0 & & \end{matrix} \right]_{\mathrm{un}} \left| \begin{matrix} m_{13} & & m_{23} & & m_{33} \\ & m_{12} & & m_{22} & \end{matrix} \right\rangle$$

$$= -(p_{33}-p_{12})[(p_{23}-p_{13})(p_{23}-p_{13}+1)]^{1/2}$$

$$\times \begin{bmatrix} [m_{13}m_{23}] & [0,-w] & [m_{12}m_{22}] \\ [10] & [00] & [10] \\ [m_{13}, m_{23}+1] & [0,-w] & [m_{12}, m_{22}+1] \end{bmatrix} \frac{K\begin{pmatrix} [m_{13}m_{23}m_{33}] \\ [m_{12}, m_{22}] \end{pmatrix}}{K\begin{pmatrix} [m_{13}, m_{23}+1, m_{33}+1] \\ [m_{12}, m_{22}+1] \end{pmatrix}}$$

One has for *any* elementary u(3) normalized reduced vertex operator that

$$\left\langle \begin{matrix} m_{13}^{(f)} & & m_{23}^{(f)} & & m_{33}^{(f)} \\ & m_{12}^{(f)} & & m_{22}^{(f)} & \end{matrix} \right\| \left[\begin{matrix} & & \Gamma_{11} & & \\ & \Gamma_{12} & & \Gamma_{22} & \\ m_{13}^{(t)} & & m_{23}^{(t)} & & 0 \\ & \gamma_{12} & & \gamma_{22} & \\ & & \gamma_{11} & & \end{matrix} \right]$$

$$\times \left| \begin{matrix} m_{13}^{(i)} & & m_{23}^{(i)} & & m_{33}^{(i)} \\ & m_{12}^{(i)} & & m_{22}^{(i)} & \end{matrix} \right\rangle$$

$$= (-1)^{\left(\phi([m_{13}^{(i)},m_{23}^{(i)}])-\phi([0,-w^{(i)}])-\phi([m_{12}^{(i)},m_{22}^{(i)}])\right)-\left(\phi([m_{13}^{(f)},m_{23}^{(f)}])-\phi([0,-w^{(f)}])-\phi([m_{12}^{(f)},m_{22}^{(f)}])\right)}$$

$$\times \left[\frac{\prod_{m=1}^{k} \prod_{\substack{l<i_m<3 \\ l\neq(i_1\cdots i_k)}} (p_{i_m3}-p_{l3})(p_{i_m3}-p_{l3}+1)}{\prod_{m=1}^{k} \prod_{\substack{l<i_m \\ l\neq(i_1\cdots i_k)}} (p_{i_m3}-p_{l3})(p_{i_m3}-p_{l3}+1)} \right]^{1/2}$$

$$\times \left[\prod_{j\neq(j_1,\ldots,j_k)} (p_{33}-p_{j2})^{\Delta_3(\Gamma)} \right] \left[\begin{matrix} [m_{13}^{(i)} m_{23}^{(i)}] & [0-w^{(i)}] & [m_{12}^{(i)} m_{22}^{(i)}] \\ [\Gamma_{12}\Gamma_{22}] & [\varpi_1\varpi_2] & [\gamma_{12}\gamma_{22}] \\ [m_{13}^{(f)} m_{23}^{(f)}] & [0-w^{(f)}] & [m_{12}^{(f)} m_{22}^{(f)}] \end{matrix} \right]$$

$$\times \langle [0,-w^{(f)}] \| \mathcal{Z}^{[\varpi_1\varpi_2]} \| [0,-w^{(i)}] \rangle \frac{K\begin{pmatrix} [m_3^{(i)}] \\ [m_2^{(i)}] \end{pmatrix} \tilde{K}\begin{pmatrix} [m_3^{(t)}] \\ [m_2^{(t)}] \end{pmatrix}}{K\begin{pmatrix} [m_3^{(f)}] \\ [m_2^{(f)}] \end{pmatrix}} \tag{4.13}$$

where

$$w^{(f)} - w^{(i)} = \Delta_1(\Gamma) + \Delta_2(\Gamma) - \Delta_1(\gamma) - \Delta_2(\gamma)$$

$$\mathcal{Z}_1^{[10]} = \partial_{13} \qquad \mathcal{Z}_0^{[0,-1]} = -z_{23}$$

$$[\varpi_1\varpi_2] = \begin{cases} [0,-(w^{(f)}-w^{(i)})] & \text{if } w^{(f)}-w^{(i)} > 0 \\ [0,0] & \text{if } w^{(f)}-w^{(i)} = 0 \\ [w^{(f)}-w^{(i)},0] & \text{if } w^{(f)}-w^{(i)} < 0 \end{cases}$$

$$\tilde{K}\begin{pmatrix} [m_3^{(t)}] \\ [m_2^{(t)}] \end{pmatrix} = \begin{cases} K\begin{pmatrix} [m_{13}^{(t)}, m_{23}^{(t)}, 0] \\ [m_{12}^{(t)}, m_{22}^{(t)}] \end{pmatrix} & \text{if } \Delta_3(\Gamma) = 0 \\[2ex] K\begin{pmatrix} [0, -m_{23}^{(t)}, -m_{13}^{(t)}] \\ [-m_{22}^{(t)}, -m_{12}^{(t)}] \end{pmatrix} & \text{if } \Delta_3(\Gamma) = 1. \end{cases}$$

When compared with the previously known result of section 4.7, we remark the following new features:

(i) There is a normalization factor, namely

$$\left[\frac{\prod_{m=1}^{k} \prod_{\substack{l<i_m<3 \\ l\neq(i_1\cdots i_k)}} (p_{i_m3}-p_{l3})(p_{i_m3}-p_{l3}+1)}{\prod_{m=1}^{k} \prod_{\substack{l<i_m \\ l\neq(i_1\cdots i_k)}} (p_{i_m3}-p_{l3})(p_{i_m3}-p_{l3}+1)} \right]^{1/2}$$

which arises because the vertex operators are not normalized. In particular, the numerator arises because u(3) vertex operators can be made up of u(2) vertex operators. Some factors may cancel in the numerator and denominator. For maximally-tied upper patterns, these factors must and do in fact all cancel out and one retrieves the results of section 4.7.

(ii) A polynomial factor, $\prod_{j\neq(j_1,\dots,j_k)}(p_{33}-p_{j2})$, now appears whenever the upper pattern is not maximally tied.

(iii) The upper and lower u(2) labels of the vertex operator labels both appear on the second line in the 9-j symbol: *for u(3) elementary tensors, the upper u(2) label refers to the intrinsic u(2) tensorial properties of the vertex operator, that is that part of the vertex operator which is coupled to the antispinor variables* (z_{13}, z_{23}) *or their derivatives.* This is an interesting result as it indicates that there can be a functional meaning to the upper label.

Unfortunately, this simple result does not generalize in a simple manner to higher order tensors as the results of the next section pertaining to totally symmetric vertex operators indicate (see nevertheless the discussions at the end of sections 4.4 and 4.5).

4.4. The totally symmetric u(3) vertex operators

The set of (multiplicity-free) totally symmetric u(3) vertex operators $[m_{13},0,0]$ is easily constructed from the u(3) fundamental vertex operators using a building-up principle: a generic unnormalized symmetric vertex operator is simply given by

$$\left\langle \begin{array}{ccccc} & & \Delta_1 & & \\ & \Delta_1+\Delta_2 & & 0 & \\ m_{13} & & 0 & & 0 \\ & & hw & & \end{array} \right\rangle_{\mathrm{un}} = (\mathrm{e}^{\mathrm{i}q_1})^{\Delta_1}(\mathrm{e}^{\mathrm{i}q_1}s_{12})^{\Delta_2}[\mathrm{e}^{\mathrm{i}q_1}(s_{12}s_{23}+s_{13}(p_2-p_3))]^{\Delta_3}.$$

It is a vertex operator with u(3) shifts given by $\Delta(\Gamma)=(\Delta_1,\Delta_2,\Delta_3)$ where

$$m_{13}=\Delta_1+\Delta_2+\Delta_3.$$

It is straightforward to verify that this vertex operator has

$$\left[\frac{\Delta_1!\Delta_2!\Delta_3!}{(\Delta_1+\Delta_2+\Delta_3)!}\prod_{i<j}^{3}\frac{(p_{i3}-p_{j3}-1)!}{(p_{i3}-p_{j3}-\Delta_j-1)!}\prod_{i<j}^{3}\frac{(p_{i3}+\Delta_i-p_{j3})!}{(p_{i3}+\Delta_i-p_{j3}-\Delta_j)!}\right]^{1/2} \tag{4.14}$$

as su(3) reduced matrix element when acting on the irrep $[m_{13}m_{23}m_{33}]$ as can be verified using elementary recoupling coefficients (Le Blanc and Hecht 1987, appendix), equations (3.10) and (4.12). It effectively reduces to (3.10) when $\Delta_3=0$.

Lower weight components for the totally symmetric tensor are relatively easy to obtain: one gets

$$\left\langle \begin{array}{ccccc} & & \Delta_1 & & \\ & \Delta_1+\Delta_2 & & 0 & \\ m_{13} & & 0 & & 0 \\ & m_{13}-k & & 0 & \\ & & m_{13}-k & & \end{array} \right\rangle_{\mathrm{un}}$$

$$
= \left[\binom{m_{13}}{k}\right]^{-1/2} \sum_{k_1+k_2+k_3=k} \binom{\Delta_1}{k_1} \left\langle \begin{array}{ccccc} & & 1 & & \\ & 1 & & 0 & \\ 1 & & 0 & & 0 \\ & 1 & & 0 & \\ & & 1 & & \end{array} \right\rangle_{\mathrm{un}}^{\Delta_1-k_1} \left\langle \begin{array}{ccccc} & & 1 & & \\ & 1 & & 0 & \\ 1 & & 0 & & 0 \\ & 0 & & 0 & \\ & & 0 & & \end{array} \right\rangle_{\mathrm{un}}^{k_1}
$$

$$
\times \binom{\Delta_2}{k_2} \left\langle \begin{array}{ccccc} & & 0 & & \\ & 1 & & 0 & \\ 1 & & 0 & & 0 \\ & 1 & & 0 & \\ & & 1 & & \end{array} \right\rangle_{\mathrm{un}}^{\Delta_2-k_2} \left\langle \begin{array}{ccccc} & & 0 & & \\ & 1 & & 0 & \\ 1 & & 0 & & 0 \\ & 0 & & 0 & \\ & & 0 & & \end{array} \right\rangle_{\mathrm{un}}^{k_2}
$$

$$
\times \binom{\Delta_3}{k_3} \left\langle \begin{array}{ccccc} & & 0 & & \\ & 0 & & 0 & \\ 1 & & 0 & & 0 \\ & 1 & & 0 & \\ & & 1 & & \end{array} \right\rangle_{\mathrm{un}}^{\Delta_3-k_3} \left\langle \begin{array}{ccccc} & & 0 & & \\ & 0 & & 0 & \\ 1 & & 0 & & 0 \\ & 0 & & 0 & \\ & & 0 & & \end{array} \right\rangle_{\mathrm{un}}^{k_3} \tag{4.15a}
$$

or, in coupled form,

$$
\left\langle \begin{array}{ccccc} & & \Delta_1 & & \\ & \Delta_1+\Delta_2 & & 0 & \\ m_{13} & & 0 & & 0 \\ & m_{13}-k & & 0 & \\ & & m_{13}-k & & \end{array} \right\rangle_{\mathrm{un}} = \left[\binom{m_{13}}{k}\right]^{-1/2} \sum_{k_3=0}^{k} \frac{(\Delta_1+\Delta_2)!}{(\Delta_1+\Delta_2-k+k_3)!}
$$

$$
\times \left[\frac{(\Delta_1+\Delta_2+1)}{(\Delta_1+\Delta_2-k+k_3+1)} \frac{1}{(k-k_3)!}\right]^{1/2}
$$

$$
\times \left[\mathcal{Z}^{[0,-(k-k_3)]}_{*}(z) \times \left\langle \begin{array}{ccc} & \Delta_1 & \\ \Delta_1+\Delta_2 & & 0 \\ & * & \end{array} \right\rangle_{\mathrm{un}}\right]^{[\Delta_1+\Delta_2-(k-k_3),0]}_{\Delta_1+\Delta_2-(k-k_3)}
$$

$$
\times \binom{\Delta_3}{k_3} \left\langle \begin{array}{ccccc} & & 0 & & \\ & 0 & & 0 & \\ 1 & & 0 & & 0 \\ & 1 & & 0 & \\ & & 1 & & \end{array} \right\rangle_{\mathrm{un}}^{\Delta_3-k_3} \left\langle \begin{array}{ccccc} & & 0 & & \\ & 0 & & 0 & \\ 1 & & 0 & & 0 \\ & 0 & & 0 & \\ & & 0 & & \end{array} \right\rangle_{\mathrm{un}}^{k_3}. \tag{4.15b}
$$

This equality states that a generic u(3) symmetric vertex operator is composed of a u(2) vertex operator of rank $[\Delta_1, \Delta_2]$ and shift (Δ_1, Δ_2), and of a u(3) vertex operator of rank $[0, 0, \Delta_3]$ and shift $(0, 0, \Delta_3)$. This observation should make it easier to decipher the corresponding normalized reduced matrix element:

$$
\left\langle \begin{array}{ccccc} m_{13}^{(f)} & & m_{23}^{(f)} & & m_{33}^{(f)} \\ & m_{12}^{(f)} & & m_{22}^{(f)} & \end{array} \right| \left[\begin{array}{ccccc} & & \Delta_1 & & \\ & \Delta_1+\Delta_2 & & 0 & \\ m_{13}^{(t)} & & 0 & & 0 \\ & m_{12}^{(t)} & & 0 & \\ & & \gamma_{11} & & \end{array} \right]
$$

$$
\times \left| \begin{array}{ccccc} m_{13}^{(i)} & & m_{23}^{(i)} & & m_{33}^{(i)} \\ & m_{12}^{(i)} & & m_{22}^{(i)} & \end{array} \right\rangle
$$

$$
\begin{aligned}
&= (-1)^{\left(\phi([m_{13}^{(i)},m_{23}^{(i)}])-\phi([0,-w^{(i)}])-\phi([m_{12}^{(i)},m_{22}^{(i)}])\right)-\left(\phi([m_{13}^{(f)},m_{23}^{(f)}])-\phi([0,-w^{(f)}])-\phi([m_{12}^{(f)},m_{22}^{(f)}])\right)} \\
&\qquad \times \left[\binom{m_{13}^{(t)}}{k} \frac{(\Delta_1+\Delta_2)!\Delta_3!}{(\Delta_1+\Delta_2+\Delta_3)!} \prod_{i=1}^{2} \frac{(p_{i3}-p_{33}-1)!}{(p_{i3}-p_{33}-\Delta_3-1)!} \right. \\
&\qquad \times \left. \prod_{i=1}^{2} \frac{(p_{i3}+\Delta_i-p_{33})!}{(p_{i3}+\Delta_i-p_{33}-\Delta_3)!} \right]^{-1/2} \sum_{k_3=0}^{k} \frac{(\Delta_1+\Delta_2)!}{(\Delta_1+\Delta_2-k+k_3)!} \\
&\qquad \times \left[\frac{(\Delta_1+\Delta_2+1)}{(\Delta_1+\Delta_2-k+k_3+1)} \frac{1}{(k-k_3)!} \right]^{1/2} \binom{\Delta_3}{k_3} \\
&\qquad \times \sum_{m_2'} U([m_{12}m_{22}][\Delta_3-k_3,0][m_{12}^{(f)}m_{22}^{(f)}] \\
&\qquad \times [\Delta_1+\Delta_2-k+k_3,0];[m_{12}',m_{22}'][m_{12}^{(t)},0]) \\
&\qquad \times \begin{bmatrix} [m_{13}^{(i)}m_{23}^{(i)}] & [0,-(w^{(i)}-\Delta_3+k_3)] & [m_{12}'m_{22}'] \\ [\Delta_1+\Delta_2,0] & [0,-(k-k_3)] & [\Delta_1+\Delta_2-k+k_3,0] \\ [m_{13}^{(f)}m_{23}^{(f)}] & [0,-(w^{(i)}-\Delta_3+k)] & [m_{12}^{(f)}m_{22}^{(f)}] \end{bmatrix} \\
&\qquad \times \langle [0,-(w^{(i)}-\Delta_3+k)] \| \mathcal{Z}^{[0,-(k-k_3)]} \| [0,-(w^{(i)}-\Delta_3+k_3)] \rangle \frac{K\begin{pmatrix}[m_3'] \\ [m_2']\end{pmatrix}}{K\begin{pmatrix}[m_3^{(f)}] \\ [m_2^{(f)}]\end{pmatrix}} \\
&\qquad \times \begin{bmatrix} [m_{13}^{(i)}m_{23}^{(i)}] & [0,-w^{(i)}] & [m_{12}^{(i)}m_{22}^{(i)}] \\ [0,0] & [\Delta_3-k_3,0] & [\Delta_3-k_3,0] \\ [m_{13}^{(i)}m_{23}^{(i)}] & [0,-(w^{(i)}-\Delta_3+k_3)] & [m_{12}'m_{22}'] \end{bmatrix} \\
&\qquad \times \sqrt{(\Delta_3-k_3)!} \langle [0,-(w^{(i)}-\Delta_3+k_3)] \| \mathcal{Z}^{[\Delta_3-k_3,0]} \| [0,-w^{(i)}] \rangle \\
&\qquad \times \left[\prod_{i=1}^{2} \frac{(p_{i3}-p_{33}-1)!}{(p_{i3}-p_{33}-\Delta_3-1)!} \right] \frac{K\begin{pmatrix}[m_3'] \\ [m_2']\end{pmatrix}}{K\begin{pmatrix}[m_3^{(i)}] \\ [m_2^{(i)}]\end{pmatrix}}
\end{aligned}
\tag{4.16}
$$

where

$$
\begin{aligned}
m_{12}^{(t)} &= m_{13}^{(t)} - k \\
m_{12}^{(f)} &= m_{12}^{(i)} + \gamma_{11} \\
m_{22}^{(f)} &= m_{22}^{(i)} + m_{12}^{(t)} - \gamma_{11}
\end{aligned}
$$

and

$$
[m_3'] = [m_{13}, m_{23}, m_{33} + \Delta_3].
$$

Although this matrix element does not present the structural simplicity of its counterpart in the previous section, the origin of the respective terms in this expression can nevertheless be traced back rather easily by comparing with equations (4.11) and (4.26).

It should be stressed that, in the limit $m_{33}^{(i)} \to -\infty$ (see the discussion at the end of the next section for the rationale underlying this limit), only the term $k_3 = k$ in the various intermediate sums survives when $k = \Delta_3$. One then has the simple result

$$\left\langle \begin{matrix} m_{13}^{(f)} & & m_{23}^{(f)} & & m_{33}^{(f)} \\ & m_{12}^{(f)} & & m_{22}^{(f)} & \end{matrix} \right| \left[\begin{matrix} & & & \Delta_1 & & \\ & & \Delta_1+\Delta_2 & & 0 & \\ & m_{13}^{(i)} & & 0 & & 0 \\ & & m_{12}^{(i)} & & 0 & \\ & & & \gamma_{11} & & \end{matrix} \right]$$

$$\times \left| \begin{matrix} m_{13}^{(i)} & & m_{23}^{(i)} & & m_{33}^{(i)} \\ & m_{12}^{(i)} & & m_{22}^{(i)} & \end{matrix} \right\rangle_{\lim m_{33}^{(i)} \to -\infty}$$

$$= \delta_{\Delta_1+\Delta_2, m_{12}^{(i)}} (-1)^{\Delta_1-\gamma_{11}} \begin{bmatrix} [m_{13}^{(i)} m_{23}^{(i)}] & [0, -w^{(i)}] & [m_{12}^{(i)} m_{22}^{(i)}] \\ [\Delta_1+\Delta_2, 0] & [0,0] & [\Delta_1+\Delta_2, 0] \\ [m_{13}^{(f)} m_{23}^{(f)}] & [0, -w^{(i)}] & [m_{12}^{(f)} m_{22}^{(f)}] \end{bmatrix}. \tag{4.17}$$

4.5. The octet vertex operators

The su(3) octet vertex operators have particular significance since they carry the adjoint representation. The operator set transforming as the octet induces the following shifts:

$[m_3] + [\Delta(\Gamma)]$	$r_1 * ([m_3] + [\Delta(\Gamma)])$	$r_2 * ([m_3] + [\Delta(\Gamma)])$
$[m_{13}+2, m_{23}+1, m_{33}]$	$[m_{23}, m_{13}+3, m_{33}]$	$[m_{13}+2, m_{33}-1, m_{23}+2]$
$[m_{13}+1, m_{23}+2, m_{33}]$	$[m_{23}+1, m_{13}+2, m_{33}]$	$[m_{13}+1, m_{33}-1, m_{23}+3]$
$[m_{13}+2, m_{23}, m_{33}+1]$	$[m_{23}-1, m_{13}+3, m_{33}+1]$	$[m_{13}+2, m_{33}, m_{23}+1]$
$[m_{13}, m_{23}+2, m_{33}+1]$	$[m_{23}+1, m_{13}+1, m_{33}+1]$	$[m_{13}, m_{33}, m_{23}+3]$
$[m_{13}+1, m_{23}, m_{33}+2]$	$[m_{23}-1, m_{13}+2, m_{33}+2]$	$[m_{13}+1, m_{33}+1, m_{23}+1]$
$[m_{13}, m_{23}+1, m_{33}+2]$	$[m_{23}, m_{13}+1, m_{33}+2]$	$[m_{13}, m_{33}+1, m_{23}+2]$
$[m_{13}+1, m_{23}+1, m_{33}+1]$	$[m_{23}, m_{13}+2, m_{33}+1]$	$[m_{13}+1, m_{33}, m_{23}+2]$
		(last line occurs twice)

One notices the characteristic double multiplicity for the shift $\Delta(\Gamma) = (1,1,1)$. The corresponding vertex operators are not mixed with the other tensors by the equivariance condition (2.6). Writing

$$t = \mathrm{e}^{\mathrm{i}(2q_1+q_2)}(s_{12}s_{23} + a([m_3])s_{13})$$

for a generic tensor t belonging to this multiplicity set, the equivariance condition (2.6) allows us to solve for the unknown coefficient $a([m_3])$:

$$s_{12}^{m_{13}-m_{23}+1}\,[(s_{12}s_{23} + a([m_3])s_{13})] = [(s_{12}s_{23} + a(r_1 * [m_3])s_{13})]\, s_{12}^{m_{13}-m_{23}+1}$$
$$s_{23}^{m_{23}-m_{33}+1}\,[(s_{12}s_{23} + a([m_3])s_{13})] = [(s_{12}s_{23} + a(r_2 * [m_3])s_{13})]\, s_{23}^{m_{23}-m_{33}+1}.$$

We find the following vertex operator

$$t = \mathrm{e}^{\mathrm{i}(2q_1+q_2)}(s_{12}s_{23} + s_{13}p_2) + X\mathrm{e}^{\mathrm{i}(2q_1+q_2)}s_{13}(p_1 + p_2 + p_3)$$

where X is an arbitrary (real) number; that is the equivariance condition (2.6) yields two independent solutions, as expected. For completeness, we give all the octet vertex operators in table 2. The multiplicity set for the shift $\Delta(\Gamma) = (1, 1, 1)$ is given by the last two entries with a particular choice for X and a particular assignment of upper patterns which we shall now justify.

Table 2. Vertex operators for the octet representation.

$$\left\langle \begin{array}{ccccc} & & 2 & & \\ & 2 & & 1 & \\ 2 & & 1 & & 0 \\ & 2 & & 1 & \\ & & 2 & & \end{array} \right\rangle_{\mathrm{un}} = \mathrm{e}^{\mathrm{i}(2q_1+q_2)}$$

$$\left\langle \begin{array}{ccccc} & & 1 & & \\ & 2 & & 1 & \\ 2 & & 1 & & 0 \\ & 2 & & 1 & \\ & & 2 & & \end{array} \right\rangle_{\mathrm{un}} = \mathrm{e}^{\mathrm{i}(2q_1+q_2)}s_{12}$$

$$\left\langle \begin{array}{ccccc} & & 2 & & \\ & 2 & & 0 & \\ 2 & & 1 & & 0 \\ & 2 & & 1 & \\ & & 2 & & \end{array} \right\rangle_{\mathrm{un}} = \mathrm{e}^{\mathrm{i}(2q_1+q_2)}s_{23}$$

$$\left\langle \begin{array}{ccccc} & & 0 & & \\ & 2 & & 0 & \\ 2 & & 1 & & 0 \\ & 2 & & 1 & \\ & & 2 & & \end{array} \right\rangle_{\mathrm{un}} = \mathrm{e}^{\mathrm{i}(2q_1+q_2)}[s_{12}^2 s_{23} - s_{12}s_{13}(p_1 - p_2 + 1)]$$

$$\left\langle \begin{array}{ccccc} & & 1 & & \\ & 1 & & 0 & \\ 2 & & 1 & & 0 \\ & 2 & & 1 & \\ & & 2 & & \end{array} \right\rangle_{\mathrm{un}} = \mathrm{e}^{\mathrm{i}(2q_1+q_2)}[s_{12}s_{23}^2 + s_{23}s_{13}(p_2 - p_3 - 1)]$$

$$\left\langle \begin{array}{ccccc} & & 0 & & \\ & 1 & & 0 & \\ 2 & & 1 & & 0 \\ & 2 & & 1 & \\ & & 2 & & \end{array} \right\rangle_{\mathrm{un}} = \mathrm{e}^{\mathrm{i}(2q_1+q_2)}[s_{12}s_{23} + s_{13}(p_2 - p_3 + 1)][s_{12}s_{23} - s_{13}(p_1 - p_2 + 1)]$$

$$\left\langle \begin{array}{ccccc} & & 1 & & \\ & 2 & & 0 & \\ 2 & & 1 & & 0 \\ & 2 & & 1 & \\ & & 2 & & \end{array} \right\rangle_{\mathrm{un}} = \mathrm{e}^{\mathrm{i}(2q_1+q_2)}[s_{12}s_{23} + s_{13}(p_2 - p_3 + 1 - F(\mathrm{p}))]$$

$$\left\langle \begin{array}{ccccc} & & 1 & & \\ & 1 & & 1 & \\ 2 & & 1 & & 0 \\ & 2 & & 1 & \\ & & 2 & & \end{array} \right\rangle_{\mathrm{un}} = \mathrm{e}^{\mathrm{i}(2q_1+q_2)}s_{13}$$

The vertex operators with u(2) upper label [11] have been identified as the su(3) generators to within a u(3) invariant factor $\mathrm{e}^{\mathrm{i}(q_1+q_2+q_3)}$. The highest weight component of this vertex operator is given by

$$\left\langle \begin{array}{ccccc} & & 1 & & \\ & 1 & & 1 & \\ 2 & & 1 & & 0 \\ & & hw & & \end{array} \right\rangle_{\mathrm{un}} = \mathrm{e}^{\mathrm{i}(2q_1+q_2)} s_{13} = -\mathrm{e}^{\mathrm{i}(q_1+q_2+q_3)} \partial_{13} = -\mathrm{e}^{\mathrm{i}(q_1+q_2+q_3)} \sigma(E_{13}).$$

The other octet vertex operators with shift $\Delta(\Gamma) = (1,1,1)$, to which must be assigned the remaining u(2) upper pattern [20], can easily be made orthogonal to the generators as follows. Using a building-up principle according to which higher rank tensor operators can be constructed from powers of the elementary tensors, we set

$$\begin{aligned} \left\langle \begin{array}{ccccc} & & 1 & & \\ & 2 & & 0 & \\ 2 & & 1 & & 0 \\ & & hw & & \end{array} \right\rangle_{\mathrm{un}} &= \left\langle \begin{array}{ccccc} & & 0 & & \\ & 0 & & 0 & \\ 1 & & 0 & & 0 \\ & & hw & & \end{array} \right\rangle_{\mathrm{un}} \left\langle \begin{array}{ccccc} & & 1 & & \\ & 1 & & 1 & \\ 1 & & 1 & & 0 \\ & & hw & & \end{array} \right\rangle_{\mathrm{un}} \\ &\quad - \left\langle \begin{array}{ccccc} & & 1 & & \\ & 1 & & 1 & \\ 2 & & 1 & & 0 \\ & & hw & & \end{array} \right\rangle_{\mathrm{un}} F(\boldsymbol{p}). \end{aligned} \tag{4.18}$$

This equality amounts to the use of a Gram–Schmidt type of orthogonalization to define the second tensor in terms of a product of elementary tensors minus the first tensor times a p-operator $F(\boldsymbol{p})$, that is a function of the momentum operators (p_1, p_2, p_3). Although the Gram–Schmidt process is, in general, not canonical since it depends on the ordering of the vectors to be orthogonalized, this result is, however, equivalent to the canonical resolution. The latter uses the fact that the space of reduced octet vertex operators for maximal lower u(2) shift is *one-dimensional*, and this not only uniquely resolves the octet multiplicity but extends to a resolution of all multiplicities for u(3) vertex operators.

The p-operator $F(\boldsymbol{p})$ is easily evaluated. When acting on the irrep $[\boldsymbol{m}_3]$, it must take the value

$$\begin{aligned} F(\boldsymbol{m}_3) = {} & U([\boldsymbol{m}_3][110][\boldsymbol{m}_3 + (111)][100]; [\boldsymbol{m}_3 + (110)]_{-}\text{-}[210]_{-}\rho = 1) \\ & \times \langle [m_{13}+1, m_{23}+1, m_{33}+1] \| \left[\begin{array}{ccccc} & & 0 & & \\ & 0 & & 0 & \\ 1 & & 0 & & 0 \end{array} \right]_{\mathrm{un}} \\ & \times \| [m_{13}+1, m_{23}+1, m_{33}] \rangle \\ & \times \langle [m_{13}+1, m_{23}+1, m_{33}] \| \left[\begin{array}{ccccc} & & 1 & & \\ & 1 & & 1 & \\ 1 & & 1 & & 0 \end{array} \right]_{\mathrm{un}} \| [m_{13}, m_{23}, m_{33}] \rangle \\ & \times [\langle [m_{13}+1, m_{23}+1, m_{33}+1] \| \left[\begin{array}{ccccc} & & 1 & & \\ & 1 & & 1 & \\ 2 & & 1 & & 0 \end{array} \right]_{\mathrm{un}} \| [m_{13}, m_{23}, m_{33}] \rangle]^{-1} \end{aligned}$$

(i) where $U(\ )$ is a u(3) recoupling coefficient in Hecht's notation (Hecht 1965, Vergados 1968) computable in terms of matrix elements of the first octet vertex operator,

(ii) and where the various quantities $\langle - \| [-] \| - \rangle$ are u(3) reduced-matrix elements of the unnormalized vertex operators. They all have been given in the previous sections except for

$$\langle [m_{13}+1, m_{23}+1, m_{33}+1] \| \begin{bmatrix} & & 1 & & \\ & 1 & & 1 & \\ 2 & & 1 & & 0 \end{bmatrix}_{\mathrm{un}} \| [m_{13}, m_{23}, m_{33}] \rangle = \left[\frac{2g(m_3)}{3} \right]^{1/2}$$

(Baird and Biedenharn 1965, Hecht 1965) where g is the su(3) Casimir invariant

$$g(m_3) = (m_{13}-m_{23})^2 + (m_{23}-m_{33})^2 + (m_{13}-m_{23})(m_{23}-m_{33}) + 3(m_{13}-m_{33}).$$

We find

$$F(p) = \left[\frac{(p_1+p_2-2p_3)(p_1-p_3+3)(p_2-p_3+2)}{2((p_1-p_2)^2+(p_2-p_3)^2+(p_1-p_2)(p_2-p_3)+3(p_1-p_3))} \right]. \qquad (4.19)$$

Our specific pattern assignment has its rationale in an expected limit property for the projective operators (Louck and Biedenharn 1973, Le Blanc and Biedenharn 1989): it states that

$$\left\langle \begin{matrix} m_{13}^{(f)} & & m_{23}^{(f)} & & m_{33}^{(f)} \\ & m_{12}^{(f)} & & m_{22}^{(f)} & \end{matrix} \right| \left[\begin{matrix} & & & \Gamma_{11} & & & \\ & & \Gamma_{12} & & \Gamma_{22} & & \\ m_{13}^{(t)} & & & m_{23}^{(t)} & & & m_{33}^{(t)} \\ & & \gamma_{12} & & \gamma_{22} & & \\ & & & \gamma_{11} & & & \end{matrix} \right]$$

$$\times \left| \begin{matrix} m_{13}^{(i)} & & m_{23}^{(i)} & & m_{33}^{(i)} \\ & m_{12}^{(i)} & & m_{22}^{(i)} & \end{matrix} \right\rangle_{m_{33}^{(i)} \to -\infty}$$

$$= \delta_{\Gamma_{12}\gamma_{12}} \delta_{\Gamma_{22}\gamma_{22}} (-1)^{\Gamma_{11}-\gamma_{11}} \begin{bmatrix} [m_{13}^{(i)} m_{23}^{(i)}] & [0-w^{(i)}] & [m_{12}^{(i)} m_{22}^{(i)}] \\ [\Gamma_{12}\Gamma_{22}] & [00] & [\gamma_{12}\gamma_{22}] \\ [m_{13}^{(f)} m_{23}^{(f)}] & [0-w^{(f)}] & [m_{12}^{(f)} m_{22}^{(f)}] \end{bmatrix}. \qquad (4.20)$$

This limit has been verified to hold for the octet vertex operators with shift $\Delta(\Gamma) = (111)$, the totally symmetric vertex operators of the last section, the vertex operators with maximally-tied upper patterns of section 4.7, as well as for all matrix elements in table 1. These results confirm the conclusion already reached in section 4.3 which states that a functional meaning can be ascribed to the hitherto abstract upper pattern: the u(3) vertex operator corresponding to the upper pattern Γ involves in its decomposition the u(2) vertex operator $\langle \Gamma \rangle$. The presence of the latter is readily made apparent by the limit (4.20).

4.6. The self-conjugate vertex operators

The next family of vertex operators which we shall briefly study is the set corresponding to the self-conjugate irreps $[2k, k, 0]$. We shall only be interested in those self-conjugate operators $[2k, k, 0]$ belonging to the multiplicity set of order $k+1$ corresponding to the shift $\Delta(\Gamma) = (k, k, k)$:

$[\boldsymbol{m}_3] + [\Delta(\Gamma)]$	$r_1 * ([\boldsymbol{m}_3] + [\Delta(\Gamma)])$	$r_2 * ([\boldsymbol{m}_3] + [\Delta(\Gamma)])$
$[m_{13}+k, m_{23}+k,$ $m_{33}+k]$	$[m_{23}+k-1, m_{13}+k+1, m_{33}+k]$	$[m_{13}+k, m_{33}+k-1,$ $m_{23}+k+1]$

Since a resolution of the multiplicity problem for the self-conjugate vertex operators with shift $\Delta(\Gamma) = (k, k, k)$ amounts in effect to a complete resolution of the u(3) multiplicity problem, it is therefore interesting to see how this multiplicity manifests itself in the present framework.

A generic tensor $[2k, k, 0]$ with shift $\Delta(\Gamma) = (k, k, k)$ can be written down as

$$\mathrm{e}^{\mathrm{i}k(2q_1+q_2)}\left[\sum_{l=0}^{k} a_l([\boldsymbol{m}_3])(s_{12}s_{23})^{k-l}s_{13}^l\right] \tag{4.21}$$

where we seek to determine the coefficients a_l. Equation (2.6) implies the relations

$$\begin{aligned}
&s_{12}^{m_{13}-m_{23}+1}\left[\sum_{l=0}^{k} a_l([\boldsymbol{m}_3])(s_{12}s_{23})^{k-l}s_{13}^l\right] \\
&\qquad = \left[\sum_{l=0}^{k} a_l(r_1 * [\boldsymbol{m}_3])(s_{12}s_{23})^{k-l}s_{13}^l\right] s_{12}^{m_{13}-m_{23}+1} \\
&s_{23}^{m_{23}-m_{33}+1}\left[\sum_{l=0}^{k} a_l([\boldsymbol{m}_3])(s_{12}s_{23})^{k-l}s_{13}^l\right] \\
&\qquad = \left[\sum_{l=0}^{k} a_l(r_2 * [\boldsymbol{m}_3])(s_{12}s_{23})^{k-l}s_{13}^l\right] s_{23}^{m_{23}-m_{33}+1}.
\end{aligned} \tag{4.22}$$

Reordering the screening charges, the coefficients $a_l([\boldsymbol{m}_3])$ are found to obey the conditions

$$\begin{aligned}
&\sum_{m=0}^{l} a_m([\boldsymbol{m}_3])\binom{k-m}{l-m}(m_{13}-m_{23}+1)^{l-m} = a_l(r_1 * [\boldsymbol{m}_3]) \\
&\sum_{m=0}^{l} a_m([\boldsymbol{m}_3])\binom{k-m}{l-m}(-1)^{l-m}(m_{23}-m_{33}+1)^{l-m} = a_l(r_2 * [\boldsymbol{m}_3]).
\end{aligned} \tag{4.23}$$

It is straightforward to verify that these two equations have the following $k+1$ independent solutions

$$[s_{12}s_{23} + m_{23}s_{13}]^{k-j}[(m_{13}+m_{23}+m_{33})s_{13}]^j \qquad 0 \le j \le k$$

thus yielding the following vertex operators

$$\left\langle \begin{array}{ccccc} & & k & & \\ & 2k-j & & j & \\ 2k & & k & & 0 \\ & & hw & & \end{array} \right\rangle_{\mathrm{un}} = \mathrm{e}^{\mathrm{i}k(2q_1+q_2)}[s_{12}s_{23}+s_{13}p_2]^{k-j}[s_{13}(p_1+p_2+p_3)]^j \tag{4.24}$$

$0 \le j \le k$, composing the multiplicity set. This multiplicity set is seen to have a structure generalizing that of the octet. The jth tensor is seen to be a product of [$k-j$ times] the second octet operator (modulo a linear combination of operators with greater js) times [j times] the first octet operator (the su(3) algebra). This tensor can thus change the lower u(2) labels $[m_{12}m_{22}]$ in the Gel'fand pattern of the initial irrep by at most $\Delta_1(\gamma) - \Delta_2(\gamma) = 2k-2j$; that is this tensor yields (after the canonical orthonormalization) the canonical set of unit tensors $[2k, k, 0]$.

4.7. *The vertex operators with maximally-tied upper patterns*

Multiplicity free vertex operators with maximally-tied upper patterns have already been discussed by Le Blanc and Biedenharn (1989). We only remark that these operators can be written in terms of screening charges as

$$\left\langle \begin{array}{ccccc} & & \Gamma_{11} & & \\ & m_{13} & & m_{23} & \\ m_{13} & & m_{23} & & m_{33} \\ & m_{13} & & m_{23} & \\ & & m_{13} & & \end{array} \right\rangle_{\mathrm{un}} = \mathrm{e}^{\mathrm{i}(m_{13}q_1+m_{23}q_2+m_{33}q_3)} s_{12}^{m_{13}-\Gamma_{11}} \tag{4.25}$$

where

$$\Gamma_{12} = m_{13} \qquad \Gamma_{22} = m_{23}.$$

One notices that, apart from the su(2) scalar factor $\mathrm{e}^{\mathrm{i}m_{33}q_3}$, these vertex operators are entirely given in terms of u(2) vertex operators and this explains why their matrix elements are computed so easily as we now briefly explain.

One obtains for this class of u(3) vertex operators the reduced normalized matrix element (Le Blanc and Biedenharn 1989)

$$\left\langle \begin{array}{ccccc} m_{13}^{(f)} & & m_{23}^{(f)} & & m_{33}^{(f)} \\ & m_{12}^{(f)} & & m_{22}^{(f)} & \end{array} \right\| \left[\begin{array}{ccccc} & & \Gamma_{11} & & \\ & \Gamma_{12} & & \Gamma_{22} & \\ m_{13}^{(t)} & & m_{23}^{(t)} & & m_{33}^{(t)} \\ & \gamma_{12} & & \gamma_{22} & \\ & & \gamma_{11} & & \end{array} \right]$$

$$\times \left| \begin{array}{ccccc} m_{13}^{(i)} & & m_{23}^{(i)} & & m_{33}^{(i)} \\ & m_{12}^{(i)} & & m_{22}^{(i)} & \end{array} \right\rangle$$

$$= (-1)^{\left(\phi([m_3^{(i)}])+\phi([m_3^{(t)}])-\phi([m_3^{(f)}])\right)-\left(\phi([m_2^{(i)}])+\phi([m_2^{(t)}])-\phi([m_2^{(f)}])\right)}$$

$$\times \begin{bmatrix} [m_{13}^{(i)} m_{23}^{(i)}] & [0 - w^{(i)}] & [m_{12}^{(i)} m_{22}^{(i)}] \\ [\Gamma_{12}\Gamma_{22}] & [0 - w^{(t)}] & [\gamma_{12}\gamma_{22}] \\ [m_{13}^{(f)} m_{23}^{(f)}] & [0 - w^{(f)}] & [m_{12}^{(f)} m_{22}^{(f)}] \end{bmatrix}$$

$$\times \left[\frac{w^{(f)}!}{w^{(i)}!w^{(t)}!}\right]^{1/2} \times \frac{K\begin{pmatrix}[m_3^{(i)}] \\ [m_2^{(i)}]\end{pmatrix} K\begin{pmatrix}[m_3^{(t)}] \\ [m_2^{(t)}]\end{pmatrix}}{K\begin{pmatrix}[m_3^{(f)}] \\ [m_2^{(f)}]\end{pmatrix}} \tag{4.26}$$

where

$$w^{(t)} = w^{(f)} - w^{(i)} > 0$$

$$\phi([m_j]) = (m_j, \rho_{su(j)}) = \frac{1}{2} \sum_{1 \le k \le j} (j - 2k + 1) m_{kj} .$$

To obtain such a simple result, we use the fact that

$$e^{i(m_{13}^{(t)} q_1 + m_{23}^{(t)} q_2)} s_{12}^{m_{13}^{(t)} - \Gamma_{11}}$$

is also the u(2) vertex operator studied in section 3: it has a well-defined action on the intrinsic u(2) labels $[m_{13} m_{23}]$ (as can be read from the 9-j symbol) and its su(3) reduced matrix element is equal in magnitude to the su(2) reduced matrix element of the corresponding u(2) vertex operator. (Indeed, the various vertex operators involved here affect only the two first entries of the partition $[m_3]$.) All reduced matrix elements then cancel out and no polynomial factors arise. The construction of such u(3) vertex operators accordingly requires the prior explicit construction of u(2) vertex operators effected in section 3.

5. Summary and conclusions

The primary purpose of this paper is to obtain analytic results for both screening operators and vertex operators in the unitary group SU(n), using the technique of holomorphic induction (generalized coherent state techniques). To accomplish this requires giving an explicit Fock space realization of the Lie algebra (sections 3 and 4.1) and the relevant equivariance condition (equation (2.6)) required to extend the vertex operators to an action on Fock modules.

For U(2), the unique screening charge is given in equation (3.6), and *all* vertex operators are constructed explicitly in equations (3.13) and (3.14).

An explicit realization of the Lie algebra of U(3) by linear differential operators acting on the space of holomorphic sections over the flag manifold is given in section 4.1. This (standard) realization is transformed explicitly ('untangled') to give the vector-coherent-state realization (section 4.3) which is adapted to the Kähler manifold (SU(2) × U(1))/SU(3). This structure is extended to Fock space and the three screening charges given in equation (4.7).

New results for the fundamental vertex operators in U(3) are given in tables 1(a), (b) and (c). An analysis of the structure of these results leads to a *single* formula encompassing *all* results for *all* elementary U(3) vertex operators (operators whose irrep labels (Young frame) consist of only 1s and 0s): this is equation (4.13). This result is new.

Complete results are also given for these special cases:

(a) The totally symmetric vertex operator $\langle m^t_{13}00\rangle$: equation (4.16)

(b) Vertex operators with maximally tied operator patterns: equation (4.26).

In these special cases, the explicit results are given in a form such that the structure is evident (K-factors and (9-j) coefficients).

To illustrate the canonical splitting of the multiplicity, the self-conjugate vertex operators (for the special case of highest weight) are given in section 4.5 (for the octet (adjoint) case) and in section 4.6 (for the general case).

Aside from direct applications in the relevant unitary group, we believe these results will be useful as models for the construction of vertex operators and screening operators in the affine unitary group using techniques developed by Bouwknegt *et al* (1990).

References

Alvarez O, Singer I M and Windey P 1990 *Nucl. Phys.* B **337** 467

Baird G E and Biedenharn L C 1965 *J. Math. Phys.* **6** 1847

Bernstein I N, Gel'fand I M and Gel'fand S I 1971 *Funct. Anal. Appl.* **5** 1

Biedenharn L C 1990 *Proc. Quantum Group Conf.* Clausthal, Germany, July 1989, to be published

Biedenharn L C and Louck J D 1968 *Commun. Math. Phys.* **8** 89

Bordemann M, Forger M and Römer H 1986 *Commun. Math. Phys.* **102** 605

Bott R 1957 *Ann. Math.* **66** 203

Bouwknegt P, McCarthy J and Pilch K 1990 *Phys. Lett.* **234B** 297

—— 1990 *Commun. Math. Phys.* **131** 25

—— 1990 *Preprint* Center for Theoretical Physics, MIT CTP 1796

Castaños O, Kramer P and Moshinsky M 1985 *J. Phys. A: Math. Gen.* **18** L493; *J. Math. Phys.* **26** 604

Castaños O, Moshinsky M and Quesne C 1985 *J. Math, Phys.* **26** 2107

Deenen J and Quesne C 1984 *J. Math. Phys.* **25** 1638, 2354

Dobaczewski J 1981 *Nucl. Phys.* A **369** 213, 237

—— 1982 *Nucl. Phys.* A **380** 1

Dotsenko V S and Fateev V A 1984 *Nucl. Phys.* B **240** 312

—— 1985 *Nucl. Phys.* B **251** 691

Feigen B L and Frenkel E V 1990 *Commun. Math. Phys.* **128** 161

Gel'fand I M and Zelevinsky A V 1985 *Soc. Math. France Astérisque* **117**

Griffiths P and Schmid W 1969 *Acta Math.* **123** 253

Guillemin V and Sternberg S 1982 *Invent. Math.* **67** 515

Hecht K T 1965 *Nucl. Phys.* **62** 1

—— 1987 *The Vector Coherent State Method and its Application to Problems of Higher Symmetries* (Berlin: Springer)

Hecht K T, Le Blanc R and Rowe D J 1987 *J. Phys. A: Math. Gen.* **20** 2241

Ito K and Kazama Y 1989 Feigin-Fuchs representations of $A_n^{(1)}$ affine Lie algebra and the associated parafermionic algebra *Preprint* University of Tokyo, UT-Komaba 89-22

Kempf G 1978 *Adv. Math.* **29** 310

Klein A and Marshalek E R 1990 Boson realizations of Lie algebras with applications to nuclear physics *Rev. Mod. Phys.* to be published

Kirillov A A 1976 *Elements of the Theory of Representations* (Berlin: Springer)

Kostant B 1970 *Lecture Notes in Mathematics* **170** (Berlin: Springer)

—— 1974 *Lecture Notes in Mathematics* **446** (Berlin: Springer)

—— 1977 *Proc. Oxford Conf. on Group Theory and Physics* Oxford

Le Blanc R and Hecht K T 1987 *J. Phys. A: Math. Gen.* **20** 4613

Le Blanc R and Rowe D J 1988 *J. Math. Phys.* **29** 758, 767

—— 1989 *J. Math. Phys.* **30** 1415
—— 1990 *J. Math. Phys.* **31** 14
Louck J D 1970 *Am. J. Phys.* **38** 3
Louck J D and Biedenharn L C 1970 *J. Math. Phys.* **11** 2368
—— 1973 *J. Math. Phys.* **14** 1336
Perelomov A M 1986 *Generalized Coherent States and Their Applications* (Berlin: Springer)
Quesne C 1986 *J. Math. Phys.* **27** 428, 869
Regge T 1958 *Nuovo Cim.* **10** 544
Rowe D J 1984 *J. Math. Phys.* **25** 2662
—— 1985 *Rep. Prog. Phys.* **48** 1419
Rowe D J, Le Blanc R and Hecht K T 1988 *J. Math. Phys.* **29** 287
Stone M 1989 *Nucl. Phys.* B **314** 557
Vergados J D 1968 *Nucl. Phys.* A **311** 681
Woodhouse N 1980 *Geometric Quantization* (Oxford: Clarendon)
Želebenko D B 1973 *Compact Lie Groups and their Representations* (Providence, RI: American Mathematical Society)